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Project-Team Café

Calcul Formel et Équations

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Activity Report

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2. Overall Objectives

Our goal is to develop computer algebra methods and software for solving functional equations, i.e. equations where the unknowns represent functions rather than numerical values, as well as to foster the use of such methods in engineering by producing the programs and tools necessary to apply them to industrial problems. We study in particular linear and nonlinear differential and (q)-difference equations, partial and ordinary.

3. Scientific Foundations

3.1. Differential ideals and D-modules

Key words: *differential systems*, *differential algebra*, *formal integrability*, *differential elimination*, *algebraic analysis*, *D-modules*, *formal integrability*, *involution*, *holonomic systems*, *control theory*.

Algorithms based on algebraic theories are developed to investigate the structure of the solution set of general differential systems. Different algebraic and geometric theories are the sources of our algorithms and making bridges between them is our challenge.

3.1.1. Formal integrability and differential elimination

Formal integrability is the first problem that our algorithms address. The idea is to *complete* a system of partial differential equation so as to be in a position to compute the *Hilbert differential dimension polynomial* or equivalently, its coefficients, the *Cartan characters*. Those provide an accurate measure of the arbitrariness that comes in the solution set (how many arbitrary functions of so many variables). Closely related is the problem of determining the *initial conditions* that can be freely chosen for having a well-posed problem (i.e. that lead to existence and uniqueness of solutions). This is possible if we can compute all the differential relations up to a given order, meaning that we cannot obtain equations of lower order by combining the existing equations in the system. Such a system is called *formally integrable* and numerous algorithms for making systems of partial differential equations formally integrable have been developed using different approaches by E. Cartan [26], C. Riquier [59], M. Janet [35] and D. Spencer [65].

Differential elimination is the second problem that our algorithms deal with. One typically wants to determine what are the lowest differential equations that vanish on the solution set of a given differential system. The sense in which *lowest* has to be understood is to be specified. It can first be order-wise, as it is of use in the formal integrability problem. But one can also wish to find differential equations in a subset of the variables, allowing the model to be reduced.

The radical differential ideal generated by a set of differential polynomials $\mathcal{S}$, i.e. the left hand side of differential equations where the right hand side is zero, is the largest set of differential polynomials that vanish on the solution set of $\mathcal{S}$. This is the object that our algorithms manipulate and for which we compute adequate representations in order to answer the above questions.

In the nonlinear case the best we can hope for is to have information outside of some hypersurface. Actually, the radical differential differential ideal can be decomposed into components on which the answers to formal integrability and eliminations are different. For each component the *characteristic set* delivers the information about the singular hypersurface together with the quasi-generating set and membership test.

Triangulation-decomposition algorithms perform the task of computing a characteristic set for all the components of the radical differential ideal generated by a finite set of differential polynomials. References for those algorithms are the book chapters written by E. Hubert [5][4]. They are based on the differential algebra developed by Ritt [60] and Kolchin [37].

The objectives for future research in the branch of triangulation-decomposition is the improvement of the algorithms, the development of alternative approaches to certain class of differential systems and the study of the intrinsic complexity of differential systems.

Another problem, specific to the nonlinear case, is the understanding and algorithmic classification of the different behaviors of interference of the locus of one component on the locus of another. The problem becomes clear in the specific case of radical differential ideals generated by a single differential polynomial. One then wishes to understand the behavior of non singular solutions in the vicinity of singular solutions. Only the case of the first order differential polynomial equations is clear. Singular solutions are either the envelope or the limit case of the non singular solutions and the classification is algorithmic [60][34].

3.1.2. Algebraic analysis

When the set $\mathcal{S}$ represent linear differential equations, we use the theory of D-modules (algebraic analysis, based on algebraic techniques such as module theory and homological algebra) as well as the formal integrability theories mentioned in the previous section (based on geometric techniques such as differential manifolds, jet spaces, involution and Lie groups of transformations). Using the duality between matrices of differential operators and differential modules, we can apply techniques that have been developed independently for those two approaches.

In addition, thanks to the works of B. Malgrange [48], V. Palamodov [52] and M. Kashiwara [36], the theory of D-modules yields new results and information about the algebraic and analytic properties of systems of linear partial differential equations, their solutions and associated geometric invariants (e.g. characteristic varieties). The above theories are becoming algorithmic thanks to the recent development of Gröbner bases [27] and involutive bases in rings of differential operators, enabling the implementation of efficient algorithms for making systems formally integrable, as well as for computing special closed-form solutions [50][51]. By using formal adjoints, it is now finally possible to algebraically study systems of linear partial differential equations, and our objectives in that field are: (i) to develop and implement efficient algorithms for computing the polynomial and rational solutions of such systems and, further, for factoring and decomposing their associated D-modules; (ii) to study the links between the algebraic and analytic properties of such systems (since the algorithmic determination of the algebraic properties yields information about the analytic properties); (iii) to apply the above algorithms to the design and analysis of linear control systems.

3.2. Groups of transformations

Key words: *symmetry*, *nonlinear differential systems*, *linear systems of partial differential equations*, *differential invariants*, *differential Galois group*, *closed-form solutions*, *variational equations*, *dynamical systems*, *Hamiltonian mechanics*, *formal integrability*.

3.2.1. Lie groups of transformations

Though not a major subject of expertise, the topic is at the crossroads of the algorithmic themes developed in the team.

The Lie group, or symmetry group, of a differential system is the (biggest) group of point transformations leaving the solution set invariant. Besides the group structure, a Lie group has the structure of a differentiable manifold. This double structure allows to concentrate on studying the tangent space at the origin, the Lie algebra. The Lie group and the Lie algebra thus capture the geometry of a differential system. This geometric knowledge is exploited to *solve* nonlinear differential systems.

The Lie algebra is described by the solution set of a system of linear partial differential equations, whose determination is algorithmic. The dimension of the solution space of that linear differential system is the dimension of the Lie group and can be determined by the tools described in Section 3.1. Explicit subalgebras can be determined thanks to the methods developed within the context of Section 3.2.2.

For a given group of transformations on a set of independent and dependent variables there exist invariant derivations and a finite set of differential invariants that generate all the differential invariants [66]. This forms an intrinsic frame for expressing any differential system invariant under this group action. This line of ideas took a pragmatic shape for computation with the general method of M. Fels and P. Olver [32] for computing the generating set of invariants. The differential algebra they consider has features that go beyond classical differential algebra. We are engaged in investigating the algebraic and algorithmic aspect of the subject.

3.2.2. Galois groups of linear functional equations

Differential Galois theory, developed first by Picard and Vessiot, then algebraically by Kolchin, associates a linear algebraic group to a linear ordinary differential equation or system. Many properties of its solutions, in particular the existence of closed-form solutions, are then equivalent to group-theoretic properties of the associated Galois group [63]. By developing algorithms that, given a differential equation, test such properties, Kovacic [38] and Singer [61] have made the existence of closed-form solutions decidable in the case of equations with polynomial coefficients. Furthermore, a generalization of differential Galois theory to linear ordinary difference equations [64] has yielded an algorithm for computing their closed-form solutions [33]. Those algorithms are however difficult to apply in practice (except for equations of order two) so many algorithmic improvements have been published in the past 20 years. Our objectives in this field are to improve the efficiency of the basic algorithms and to produce complete implementations, as well as to generalize them and their building blocks to linear partial differential and difference equations.

An exciting application of differential Galois theory to dynamical systems is the Morales-Ramis theory, which arose as a development of the Kovalevskaya-Painlevé analysis and Ziglin's integrability theory [70][71]. By connecting the existence of first integrals with branching of solutions as functions of complex time to a property of the differential Galois group of a variational equation, it yields an effective method of proving non-integrability and detecting possible integrability of dynamical systems. Consider the system of holomorphic differential equations

$$\dot{x} = v(x), \qquad t \in \mathbb{C}, \quad x \in M, \tag{1}$$

defined on a complex n-dimensional manifold M. If $\varphi(t)$ is a non-equilibrium solution of (1), then the maximal analytic continuation of $\varphi(t)$ defines a Riemann surface Γ with t as a local coordinate. Together with (1) we consider its variational equations (VEs) restricted to $T_\Gamma M$, i.e.

$$\dot{\xi} = T(v)\xi, \qquad \xi \in T_\Gamma M.$$

We can decrease the order of that system by considering the induced system on the normal bundle $N := T_\Gamma M/T\Gamma$ of Γ:

$$\dot{\eta} = \pi_\star(T(v)\pi^{-1}\eta), \qquad \eta \in N$$

where $\pi : T_\Gamma M \to N$ is the projection. The system of $s = n - 1$ equations obtained in that way yields the so-called normal variational equations (NVEs). Their monodromy group $\mathcal{M} \subset \mathrm{GL}(s, \mathbb{C})$ is the image of the fundamental group $\pi_1(\Gamma, t_0)$ of Γ obtained in the process of continuation of the local solutions defined in a neighborhood of t_0 along closed paths with base point t_0. A non-constant rational function $f(z)$ of s variables $z = (z_1, ..., z_s)$ is called an integral (or invariant) of the monodromy group if $f(g \cdot z) = f(z)$ for all $g \in \mathcal{M}$. In his two fundamental papers [70][71], Ziglin showed that if (1) possesses a meromorphic first integral, then $\mathcal{M}$ has a rational first integral. Ziglin found a necessary condition for the existence of a maximal number of first integrals (without involutivity property) for analytic Hamiltonian systems, when $n = 2m$, in the language of the monodromy group. Namely, let us assume that there exists a non-resonant element $g \in \mathcal{M}$. If the Hamiltonian system with m degrees of freedom has m meromorphic first integrals $F_1 = H, ..., F_m$, which are functionally independent in a connected neighborhood of Γ, then any other monodromy matrix $g' \in \mathcal{M}$ transforms eigenvectors of g to its eigenvectors.

There is a problem however in making that theory algorithmic: the monodromy group is known only for a few differential equations. To overcome that problem, Morales-Ruiz and Ramis recently generalized Ziglin's approach by replacing the monodromy group $\mathcal{M}$ by the differential Galois group $\mathcal{G}$ of the NVEs. They formulated [49] a new criterion of non-integrability for Hamiltonian systems in terms of the properties of the connected identity component of $\mathcal{G}$: if a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighborhood of the analytic curve Γ, then the identity component of the differential Galois group of NVEs associated with Γ is Abelian. Since $\mathcal{G}$ always contains $\mathcal{M}$, the Morales-Ramis non-integrability theorem always yields stronger necessary conditions than the Ziglin criterion.

When applying the Morales-Ramis criterion, our first step is to find a non-equilibrium particular solution, which often lies on an invariant submanifold. Next, we calculate the corresponding VEs and NVEs. If we know that our Hamiltonian system possesses k first integrals in involution, then we can consider VEs on one of their common levels, and the order of NVEs is equal to $s = 2(m - k)$ [23][24]. In the last step we have to check if the identity component of $\mathcal{G}$ is Abelian, a task where the tools of algorithmic Galois theory (such has the Kovacic algorithm) become useful. In practice, we often check only whether that component is solvable (which is equivalent to check whether the NVEs have Liouvillian solutions), because the system is not integrable when that component is not solvable.

Our main objectives in that field are: (i) to apply the Morales-Ramis theory to various dynamical systems occurring in mechanics and astronomy; (ii) to develop algorithms that carry out effectively all the steps of that theory; (iii) to extend it by making use of non-homogeneous variational equations; (iv) to generalize it to various non-Hamiltonian systems, e.g. for systems with certain tensor invariants; (v) to formulate theorems about partial integrability of dynamical systems and about real integrability (for real dynamical systems) in the framework of the Morales-Ramis theory;

3.3. Mathematical web services

Key words: *Computer algebra*, *formula databases*, *deductive databases*, *communication*, *Web services*, *OpenMath*, *MathML*.

The general theme of this aspect of our work is to develop tools that make it possible to share mathematical knowledge or algorithms between different software systems running at arbitrary locations on the web.

Most computer algebra systems deal with a lot of non algorithmic knowledge, represented directly in their source code. Typical examples are the values of particular integrals or sums. A very natural idea is to group this knowledge into a database. Unfortunately, common database systems are not capable to support the kind of mathematical manipulations that are needed for an efficient retrieval (doing pattern-matching, taking into account commutativity, etc.). The design and implementation of a suitable database raise some interesting problems at the frontier of computer algebra. We are currently developing a prototype for such a database that is capable of doing some deductions. Part of our prototype could be applied to the general problem of searching through mathematical texts, a problem that we plan to address in the near future.

The computer algebra community recognized more than ten years ago that in order to share knowledge such as the above database on the web, it was first necessary to develop a standard for communicating mathematical objects (via interprocess communication, e-mail, archiving in databases). We actively participated in the definition of such a standard, OpenMath (partly in the course of a European project). We were also involved in the definition of MathML by the World Wide Web Consortium. The availability of these two standards are the first step needed to develop rich mathematical services and new architectures for computer algebra and scientific computation in general enabling a transparent and dynamic access to mathematical components. We are now working towards this goal by experimenting with our mathematical software and emerging technologies (Web Services) and participating in the further development of OpenMath.

4. Application Domains

4.1. Panorama

We have applied our algorithms and programs for computing differential Galois groups to determine necessary conditions for integrability in mechanical modeling and astronomy (see 6.1). We also apply our research on partial linear differential equations to control theory, for example for parameterization and stabilization of linear control systems (see 6.6 and 6.7).

5. Software

5.1. Maple package diffalg

Participant: Evelyne Hubert.

Key words: *nonlinear differential systems*, *differential elimination*, *analysis of singular solutions*, *differential algebra*, *triangulation-decomposition algorithms*.

The *diffalg* library has been part of the commercial release of Maple since Maple V.5 and has evolved up to Maple 7. The first version of the library is due to F. Boulier, Université de Lille. The library implements a triangulation-decomposition algorithm for polynomially nonlinear systems and tools for the analysis of singular solutions.

A new release as a package will be available in early 2004. The high point of the new release is the implementation of algorithms for differential polynomial rings where derivations satisfy nontrivial commutation rules. That follows the investigations on the algebra of the differential invariants but bears already on several other fields of applications in mathematics. This new version also incorporates a specific treatment of parameters as well as improved algorithms for higher degree polynomials.

5.2. Library OreModules of Mgfun

Participants: Frédéric Chyzak, Alban Quadrat [correspondent], Daniel Robertz.

Key words: *Effective algebraic analysis*, *Ore algebras*, *Gröbner basis*, *linear systems*.

A library *OreModules* of MGFUN has been developed on some effective aspects of algebraic analysis. It is dedicated to the study of under-determined linear systems over some Ore algebras (e.g. ordinary differential equations, partial differential equations, differential time-delay equations, discrete equations) and their applications in mathematical physics (e.g. research of potentials, computations of the field equations and the conservation laws) and in linear control theory (e.g. controllability and flatness of multidimensional systems with varying coefficients, parameterizability, motion planning). The main novelty of *OreModules* is to use the recent development of the Gröbner basis over some Ore algebras (non-commutative polynomial rings) in order to effectively check some properties of module theory (e.g. torsion/torsion-free/reflexive/projective/free modules) and homological algebra (e.g. free resolutions, split exact sequences, duality, extension functor, dimensions).

A library of examples is in development but several examples (two pendulum mounted on a car, a time-varying system of algebraic equations, a wind tunnel model, a two reflector antenna, an electric transmission line, Einstein equations, Lie-Poisson structures) are already freely available with *OreModules*.

5.3. Library libaldor

Participant: Manuel Bronstein.

Key words: *Aldor*, *standard*, *data structures*, *arithmetic*.

The LIBALDOR library, under development in the project for several years, became the standard ALDOR library, distributed with the compiler back in 2001. During 2003, we have added new data structures and enhanced the I/O features in order to facilitate the porting to LIBALDOR of external ALDOR projects. The new library, version 1.0.2 is expected to be officially released in December 2003.

5.4. Library Algebra

Participants: Manuel Bronstein [correspondent], Marc Moreno-Maza.

Key words: *computer algebra*, *linear algebra*, *commutative algebra*, *polynomials*.

The ALGEBRA library, written in collaboration with Marc Moreno-Maza (UWO, London, Ontario), keeps on being expanded, with new linear algebra modules as well as additional functionalities for lazy power series. Starting with the 1.0.2 release, expected December 2003, the ALGEBRA library is being distributed together with the ALDOR compiler.

5.5. Library Σ^{it}

Participant: Manuel Bronstein.

Key words: *computer algebra*, *differential equations*, *difference equations*, *systems*.

The Σ^{it} library contains our algorithms for solving functional equations. An important milestone achieved in 2003 is a complete implementation of our customization of the algorithm of van Hoeij & Weil [68] for computing invariants of differential Galois groups, as well as an extension for semi-invariants. Based on that feature, available in the 2003 version of the interactive server `Bernina`, we have implemented a complete solver for linear ordinary differential equations of order 3, and made that solver publicly available as a web service at http://www-sop.inria.fr/cafe/Manuel.Bronstein/sumit/bernina_demo.html.

6. New Results

6.1. Integrability analysis of dynamical systems

Participants: Maria Przybylska, Jacques-Arthur Weil.

Key words: *Hamiltonian systems*, *non-Hamiltonian systems*, *integrability*, *real integrability*.

Together with A. Maciejewski (Zielona Góra), we studied in detail the Morales-Ramis theory from the point of view of its applications to integrability analysis. We used it to study the integrability of the following systems: a rigid satellite moving under influence of geomagnetic field [12], the restricted two-body problems in constant curvature spaces [43], the generalized two fixed centers problem [44], the Suslov problem [45], the generalized spring-pendulum problem, the heavy top problem [42] and the class of Gross-Neveu systems [46]. Almost all the above systems depend on parameters. We proved non-integrability of all of them for almost all values of their parameters except for some distinguished values or families of parameters. For those distinguished values, either the systems are integrable (i.e. there are known first integrals) or the Morales-Ramis theory does not yield a final answer. In those cases we used a version of the Morales-Ramis theory with variational equations of higher order. We found an example of a system [47] that shows that the Morales conjecture about the possibility of formulating sufficient integrability conditions in the language of the differential Galois group of the variational equations of higher orders is not correct. We have also shown [45] that in some cases it is possible to apply the Morales-Ramis theory to non-Hamiltonian systems. Finally, we computed [44] differential Galois groups of equations with infinitely many singular points (their number depends on the parameters of the problem). Our calculations show that the technique of symmetric powers is very useful for checking whether differential Galois groups are Abelian.

The Morales-Ramis theory concerns only complex integrability. We showed how combining some results of that theory and the Ziglin theory makes it possible to prove real non-integrability for real dynamical systems [45][42]. Thanks to the expertise that we have gathered during our studies of particular dynamical systems, we can now try to formulate certain general results about integrability and about the Morales-Ramis theory.

6.2. Algorithms for nonlinear differential systems

Participants: Evelyne Hubert, Nicolas Le Roux.

We study the effective and efficient computation of power series solutions of partial differential systems.

For a finite set of differential polynomials, triangulation-decomposition algorithms output a finite set of regular differential chains. The set of zeros of the original system is the union of the non singular zeros of the output regular differential chains. For each regular differential chain we can determine the generic initial conditions that lead to existence and uniqueness of a power series solution. The proof of this formal integrability of regular differential chains is constructive and leads to a first algorithm to compute the power series. The coefficients of the power series are obtained by successive derivations of the differential polynomials in the regular differential chains.

We proposed another approach to the computation of the power series solution of regular differential chains. The method is of Newton type: at each step, the linearisation of the system is used to double the size of the approximation so far. More precisely, if the power series is known up to order p, we use recurrence relations induced by the linearisation of the system to extend the power series solution up to order $2p - r$, where r is the order of the system.

Our results were presented at the conference ISSAC 2003 in Philadelphia [19].

We seek to further improve the computation performance by using modular computation followed by a lifting of Hensel type. For that, we must work out a bound on the coefficients of the power series solutions.

6.3. Algebraic structures for differential invariants

Participant: Evelyne Hubert.

The basic assumption of classical differential algebra and differential elimination is that the derivations do commute, which is the standard case arising from systems of partial differential equations. We developed the a generalization of the theory to the case where the derivations satisfy nontrivial commutation rules. That situation arises, for instance, when we consider a system of equations on the differential invariants of a Lie group action.

It had already been noted [31][41] that computations for differential elimination could be made easier by using derivations that conceal some knowledge of the nature of problem, like for instance geometry in the case that is of interest to us. We provided firm foundations and a general framework for the validity of such computations.

The results are written in [22] and are now submitted for publications. We provided an implementation of a triangulation-decomposition algorithm for that new case as an extension of the *diffalg* Maple library (cf. 5.1).

6.4. Algorithms for solving linear ordinary equations and systems

Participants: Manuel Bronstein, Damien Jamet, Rohit Shrivastava, Jacques-Arthur Weil.

Key words: *differential equations*, *difference equations*, *Galois group*, *systems*, *algebraic curves*, *algebraic functions*, *integration*.

6.4.1. Linear ordinary differential systems:

The reduction developed in 2002 with B.M. Trager (IBM Research) has been applied to the lazy Hermite integration of algebraic functions, which has been implemented in MAPLE by Damien Jamet. A new improvement, computing a good initial basis of integral elements via Newton sums, was discovered and implemented, yielding very efficient results for integration. This work has been presented at the MEGA'2003 conference [17] and has been submitted to the *Journal of Symbolic Computation*. In addition, since the 2002 implementation of the genus computation has shown that computing the local exponents at singularities is a significant computing step, a hybrid symbolic/numeric approach that approximates those exponents using LAPACK has been implemented in ALDOR by Rohit Shrivastava.

6.4.2. Linear ordinary differential operators:

In collaboration with E. Compoint [29] (Univ. Lille I), we have developed a theory and algorithms to characterize the cases when a differential operator is reducible over an algebraic extension of its constant field. The material elaborated in this work is a step towards a general method for determining differential Galois groups of differential operators which, in turn, yields structural properties of their solutions.

In addition, we have also completed our implementation in the Σ^{it} library (5.5) of a slight variant of the algorithm of Hoeij & Weil [68] for computing invariants of differential Galois groups, as well as an extension for semi-invariants. Using a new Hessian-based test for finding invariants that factor linearly, we then produced a complete solver for linear ordinary differential equations of order 3, roughly along the lines described by van Hoeij & al. [67]. Our solver, the first distributed software solving that problem, is also available as a web service at http://www-sop.inria.fr/cafe/Manuel.Bronstein/sumit/bernina_demo.html.

6.5. Algorithms for solving partial differential equations

Participants: Manuel Bronstein, José-Luis Martins, Jacques-Arthur Weil, Min Wu.

Key words: *Partial differential equations*, *partial difference equations*, *polynomial solutions*, *rational solutions*, *D-modules*, *factorization*, *decomposition*.

6.5.1. Factorization of linear systems of partial functional equations:

We continued our work towards developing effective algorithms and programs to factor systems of linear partial differential or (q)-difference equations with finite-dimensional solution spaces (called fLPDEs). The Li-Schwarz-Tsarev algorithm [40] is a generalization of Beke's factorization algorithm [25] from linear ODEs to fLPDEs. That generalization uses Beke spaces and relies on the Li-Schwarz algorithm [39] to compute rational solutions of Riccati-like partial differential equations. We are investigating instead the D-module interpretation of the Beke algorithm [63], which allows us to express not only Beke's algorithm, but also the Plücker relations and the eigenring decomposition method [62].

This year, we generalized Beke's approach to factoring finite-dimensional D-modules where D is a ring of either partial differential or difference operators. This improves the efficiency of the Li-Schwarz-Tsarev algorithm, since we solve one system of Riccati equations per candidate dimension d of the factor, as opposed to up to 2^d systems in their case. In addition, this yields a similar algorithm for factoring systems of linear partial difference equations since Li has recently been able to solve Riccati-like equations in that setting. This work is being written up in Min Wu's dissertation and will be submitted for publication in 2004.

6.5.2. Rational solutions of integrable models related to mathematical physics:

The equation $a_{xxx} + 4ba_x + 2b_xa + 2b_t = 0$ appears in Jules Drach's thesis. There, he proposes to generalize Galois theory to transcendental equations (e.g. nonlinear differential equations) and his equation plays a central role. The same equation unexpectedly appears in the classification of integrable systems of PDEs with a two dimensional solution space, when one asks that the differential Galois group should be unimodular. For these reasons, the description of the rational solutions of Drach's equation is of mathematical importance. Drach's equation reduces to the KdV model of long waves in shallow water by taking $a = b$. In the same way, by imposing $2b_x - 3aa_x - 3a_t = 0$ one recovers Boussinesq's model. More generally, most of the commonly studied integrable models in mathematical physics appear to result from Drach's equation by a suitable reduction. For each such model, there are papers obtaining a description of the corresponding rational solutions. By investigating the algebraic and differential properties of Drach's equation, we've succeeded in producing a scheme for describing it's rational solutions. For specific choices of reductions $F(a, b) = 0$ this scheme delivers an algorithm for describing the rational solutions of the corresponding equations (KdV, Boussinesq, etc.). Rational solutions of Drach's equation are in turn useful as they produce integrable linear systems of partial differential equations that can be used to test the algorithms described earlier in this section.

6.6. Linear systems over Ore algebras and applications

Participants: Alban Quadrat, Daniel Robertz.

Key words: *Algebraic analysis*, *(under-/over-determined) linear systems*, *Ore algebras*, *multidimensional systems*, *Gröbner basis for non-commutative polynomial rings*.

In [18] and another submission [28], we study some structural properties of linear systems over Ore algebras (e.g. systems of ordinary differential equations, systems of differential time-delay equations, systems of partial differential equations, systems of difference equations). Using the recent development of Gröbner basis over some Ore algebras (i.e. non-commutative polynomial rings), we show how to make effective some important concepts of homological algebra (e.g. free resolutions, split exact sequences, duality, extension functor, dimensions). Then, using these results, we obtain some effective algorithms which check the different structural properties of under-determined linear systems over Ore algebras, developed using module theory (e.g. torsion/torsion-free/reflexive/projective/free modules). In particular, we explain why these properties are related to the possibility to successively parameterize all solutions of a system and its parameterizations. Moreover, we show how these properties generalize the well-known concepts of primeness, developed in the literature of multidimensional systems, to systems with varying coefficients. Then, using a dictionary between the structural properties of under-determined linear systems over Ore algebras and some concepts of linear control theory, we show that the previous algorithms allow us to effectively check whether or not a linear control system over certain Ore algebras is (weakly, strongly) controllable, parameterizable, flat, π-free... Finally, using the implementation of all these algorithms in *OreModules* (5.2), these results are illustrated on different systems (e.g. two pendulum mounted on a car, a time-varying system of algebraic equations, a wind tunnel model, a two reflector antenna, an electric transmission line, Einstein equations, Lie-Poisson structures). Let us remark that the problem of parameterizing all the solutions of linear controllable multidimensional systems has been extensively studied by the school of M. Fliess in France and J. C. Willems in the Netherlands [53]. Hence, our allow us to effectively answer that problem for a large classes of systems considered in the literature.

In [13], we explain how the module-theoretic approach to linear control systems, developed in the literature after the pioneering work of R. E. Kalman, is dual to the behavioral approach developed by the school of J. C. Willems [53]. Moreover, we show the relationship between the possibility to obtain scalar decoupled quantities in the system and the lack of parameterization of all solutions of the system for general linear systems.

For linear systems of partial differential equations, we have shown [54] how to use the previous results in order to study some variational problems. In particular, we show how the Lagrangian given by the variational problem links the sequence formed by the differential operator describing the system and its parameterization and the sequence formed by their formal adjoints (Poincaré duality). Using this result, we give different new equivalent forms for the optimal system (with or without the Lagrangian multipliers). These results are illustrated on examples coming from linear elasticity, electromagnetism and linear quadratic optimal problem.

6.7. Algebraic analysis approach to infinite-dimensional linear systems

Participant: Alban Quadrat.

Key words: *Systems of partial differential equations or differential time-delay equations*, *internal/strong/simultaneous/robust/optimal stabilization*, *algorithms*, *module theory*.

In [15][16], we study *infinite-dimensional linear systems* [30], namely some classes of systems of partial differential equations (e.g. wave, heat, Euler-Bernoulli equations), differential time-delay systems (e.g. transport equation, transmission lines), fractional differential systems... usually encountered in mathematical physics, within an algebraic analysis framework. Let us remark that until today, no one has been able to show how to incorporate the initial/boundary conditions of the system in the classical differential module (D-module) approach. In order to face this problem, using symbolic calculus (i.e. Laplace transform and symbolic integration), we transform an invariant infinite-dimensional linear system into a transfer matrix, i.e. into functional relations between the inputs and the outputs of the system. Then, using *the fractional representation approach to systems* [69], we show in [15][16] how to develop a module-theoretic approach to infinite-dimensional linear systems. An important issue in control theory is to stabilize an unstable system (e.g. a system with an infinite number of unstable poles) by adding a controller in feedback to the system (internal stabilization). In [15][16], using the module-theoretic approach to infinite-dimensional linear systems, we obtain general necessary and sufficient conditions for the existence of (weakly) left/right/doubly coprime factorizations and for internal stabilizability. Up to our knowledge, these necessary and sufficient conditions are the most general ones in the literature and give an answer to a question of M. Vidyasagar, B. Francis and H. Schneider about the links between internal stabilizability and the existence of doubly coprime factorizations [69]. Finally, we characterize all the algebras such that one of the previous properties is always satisfied and we develop some algorithms in order to check these different properties.

In [14][20][21] and a submitted paper [56], using the algebraic concepts of lattices and fractional ideals, we show how to generalize the well-known *Youla-Kučera parameterization of all stabilizing controllers* [69] for systems which do not necessarily admit coprime factorizations. This new parameterization allows us to rewrite the problem of finding the optimal stabilizing controllers (for a certain norm) as affine, and thus, convex problems. In particular, we solve the well-known conjecture of Z. Lin on the equivalence between internal stabilizability and the existence of doubly coprime factorizations for multidimensional systems.

We have also shown [58] how the algebraic concept of *stable range*, introduced in algebra by H. Bass, plays a central role in the *strong stabilization problem* (stabilization of a plant by means of a stable controller) [69]. In particular, we exhibit the particular form of certain stabilizing controllers in which the size of the unstable part depends on the stable range of the system. This result allows us to prove that any multi-input multi-output stabilizable system over $H_\infty(\mathbb{C}_+)$ is strongly stabilizable. Let us remark that this result answers a question asked in the problem 21 of "Open Problems on the Mathematical Theory of Systems", 2002 MTNS Problem Book.

We have also shown [55] how the operator-theoretic approach to linear systems [30] is dual to the module-theoretic approach [14]. This new theory plays a similar role as the behavioral approach to multidimensional linear systems developed by J. C. Willems and his school [53]. In particular, using the algebraic concept of fractional ideals, we exhibit the precise domain and graph of an internal stabilizable system with a single-input and single-output. This result generalizes all the ones known in the literature.

All these results have been summarized in some lecture notes for the summer school "International School in Automatic Control of Lille" entitled "Control of Distributed Parameter Systems: Theory and Applications", organized by M. Fliess, 02-06/09/02, Ecole Nationale de Lille (France). They will appear [57] with the proceedings of that summer school.

7. Contracts and Grants with Industry

7.1. Waterloo Maple Inc.

Participant: Manuel Bronstein.

Through this one-year contract, WMI (makers of the computer algebra system MAPLE) supports the collaboration between CAFÉ and Prof. Abramov's group at the Russian academy of science, by paying for travel and living expenses. In exchange, WMI gets early reports on the results of that collaboration as well as a faster track between algorithmic developments and their distribution as part of MAPLE. A prolongation for 2004 is awaiting signature.

8. Other Grants and Activities

8.1. National initiatives

A. Quadrat is a member of the working group "Systèmes à Retards" of the *GdR Automatique*.

8.2. European initiatives

8.2.1. OpenMath

Participants: Stéphane Dalmas, Marc Gaëtano.

CAFÉ continues its participation in the *OpenMath* (IST-2000-28719) Thematic Network which is a follow on from the earlier ESPRIT Project. The network's main activities are to organize workshops bringing together people working on OpenMath from around the world, to provide a continued focus-point for the development of the OpenMath Standard, to facilitate European participation in the W3C Math Working group, to coordinate the development of OpenMath and MathML tools, to coordinate the development of OpenMath and MathML applications and to disseminate information about OpenMath and MathML.

The current membership of the network is NAG Ltd (UK, coordinator), the University of Bath (UK), Stilo Technology Ltd (UK), INRIA, the University of St Andrews (UK), the Technical University of Eindhoven (Netherlands), Springer Verlag (Germany), the University of Nice Sophia Antipolis, ZIB (Germany), Explo-IT Research (Italy), RISC (Austria), German Research Center for Artificial Intelligence (Germany), and the University of Helsinki (Finland).

8.2.2. PAI Alliance

Our PAI *Alliance* project *moving frames and differential systems* continued to support the collaboration between E. Hubert and Elisabeth Mansfield (University of Canterbury at Kent) during 2003.

8.2.3. PAI Polonium

K. Avratchenkov (MISTRAL), P.A. Bliman (SOSSO) and A. Quadrat (CAFÉ) have a collaboration with Prof. K. Galkowski's group at the University of Zielona Góra (Poland) within the framework of an exchange research program PAI *Polonium* entitled "Theory and applications of n-dimensional systems, delay systems and iterative learning control" started in 2003.

8.2.4. European training site

Under the supervision of A. Quadrat, the project "Computational methods in linear control systems" of Daniel Robertz, PhD student at the University of Aachen (Germany), has been granted from the Control Training Site (CTS).

8.3. Other international initiatives

8.3.1. France-Canada research fund

This was the second and last year of our supported collaboration with the Ontario Research Center for Computer Algebra on the topic of *extension of classical differential algebra and related software*. A meeting between Greg Reid, Evelyne Hubert and Nicolas Le Roux was held in Philadelphia at the occasion of the ISSAC'2003 conference. Gregory Reid plans to spend his sabbatical term in our project in the fall of 2004.

8.3.2. Liapunov Institute

Our collaboration with Prof. S.A. Abramov (Moscow) continues to be supported by the Liapunov institute within the project *Hypergeometric Computer Algebra*. During 2003, we improved the `LinearFunctionalSystems` MAPLE package, which implements the equation solvers that we developed in the previous years, and performed some extensive benchmarks showing its superiority to all the other known solvers. We also explained in a report the differences with the other algorithms and the reasons for the superior benchmark results [8]. An ECO-NET proposal has been submitted to join Prof. M. Petkovšek (Ljubljana) to this collaboration.

8.3.3. China

Following the termination of our PRA with Z. Li (Academia Sinica), our collaboration continues with the co-direction of the thesis of Min Wu, who is alternating 6-month stays in our project and in Beijing.

8.4. International networks and working groups

Stéphane Dalmas was a member of the W3C Math Working Group. This group was responsible for defining MathML, an XML application for describing mathematical notation and capturing both its structure and content. The goal of MathML is to enable mathematics to be served, received, and processed on the World Wide Web, just as HTML has enabled this functionality for text.

In 2003 the Working Group released MathML 2.0, second edition that became a W3C Recommendation in October. The charter of the Math Working Group expired in September and a Math Interest Group will be created to continue the work of maintaining existing documents, working with other W3C groups and provide general support on MathML and mathematics on the Web.

8.5. Visiting scientists

8.5.1. Europe (CE)

Under the sponsorship of the Control Training Site (Sect. 8.2.4), Daniel Robertz visited our project for three months (February-April 2003) in order to collaborate on the *OreModules* library (see 5.2).

Within the framework of our PAI Alliance (Sect. 8.2.2), Peter Clarkson and Elizabeth Mansfield (U. of Canterbury at Kent, UK) visited our project for one week in June 2003 to work with E. Hubert on moving frames and differential systems.

Ralf Hemmecke (RISC Linz, Austria) visited our project for one week in November 2003 in order to start porting his Calix software on top of our LIBALDOR and ALGEBRA libraries (see 5.3). A PAI Amadeus proposal for 2005 is being prepared.

8.5.2. Europe (outside CE)

Yuri Rappoport (CC RAS, Moscow) gave a talk in the CAFÉ seminar in February 2003.

Andrzej Maciejewski (Zielona Góra, Poland) visited our project for three 1-week periods in March, October and December, to work with M. Bronstein, M. Przybylska and J-A. Weil on the integrability of dynamical systems.

As part of our Liapunov project (see 8.3.2), S.A. Abramov and D.E. Khmelnov (CC RAS, Russia) visited our project for two 10-day periods in May and November 2003.

8.5.3. Outside Europe

As part of our joint library development for the ALDOR compiler (see 5.3), Stephen Watt (UWO, Canada) visited our project for two 1-week periods in June and December 2003, and Marc Moreno Maza (UWO, Canada) for one week in June 2003.

Lourdes Juan (Texas Tech University) visited our project for one week in June 2003 and presented her results on the inverse problem in differential Galois theory in the CAFÉ seminar.

Shiva Shankar (Chennai Mathematical Institute, India) visited our project for 1 week in July 2003 to work with A. Quadrat on behavioral control theory, which he presented in the CAFÉ seminar.

Kathy Horadam (Melbourne, Australia) presented differential cryptanalysis in the CAFÉ seminar in September 2003.

9. Dissemination

9.1. Leadership within scientific community

- Evelyne Hubert was a member of the program committee of the ISSAC'2003 conference.
- The goals of the association *femmes & mathématiques* include promoting scientific studies and the image of women in academic careers as well as providing occasions for female mathematicians to meet. The association works in collaboration with governmental institutions and other association.
  Evelyne Hubert is member of the council of the association. As such she participates to the monthly meetings of the council to propose, discuss and undertake the actions of the association. Her major contribution this year is chairing the organization of the *forum des jeunes mathématiciennes* to be held in Paris in January 2004.
  The forum is a biennial francophone conference where senior and junior female mathematicians can meet so as to favor mentoring, role model and identification. Beside the scientific talks, the forum stages debates on subject relating to science and education and talks given by researchers in humanities working on women studies. The present forum focuses on mathematics for (natural) sciences. It has received the financial support of both INRA and INRIA.
- In collaboration with Sylvie Poupinel (ACI Grid), Evelyne Hubert launched two sessions of *Séminaires Croisés*, a rather unique initiative of global scientific animation within the site of INRIA Sophia Antipolis. Talks given by second year PhD students gather several teams within a thematic day. The *Séminaires Croisés* provide opportunities for students to present formally their work in a reasonably relaxed atmosphere. It is an occasion for all to interact constructively with other teams on the site. In 2003, there has been 6 thematic days with a total of 24 talks, a fair success.
- Evelyne Hubert is a member of the Committee of Doctoral Studies at INRIA Sophia (*comité du suivi doctoral*) chaired by Thierry Vieville. The committee evaluates the documents for new PhDs and postdoctoral fellows and serves for advice in the course of the doctoral studies. The committee members act as moderators when required by supervisors or students and possibly investigate after decision of the committee. The committee is also in charge of ranking the candidacies for doctoral and postdoctoral grants.
- Evelyne Hubert is a member of the *Colors* Committee chaired by Rose Dieng. The committee is in charge of selecting project of collaborations of INRIA teams with local industrial or academic actors.
- In the framework of the Graduiertenkolleg of the University of Aachen (Germany), A. Quadrat was invited to organize a summer school on the effective algebraic aspects of linear control theory (theory, applications and packages). The summer school, entitled “Introduction to Algebraic Control Theory: from finite to infinite-dimensional systems”, which has been held at Otzenhausen in September 2003, had 30 attendees (graduate students, lecturers and professors).
- A. Quadrat organized the unique invited session at the Workshop on Time-Delay Systems, IFAC, which has been held at INRIA Rocquencourt in September 2003. The invited session, entitled “Algebraic and geometric approaches to linear differential time-delay systems” contained 6 papers on different effective algebraic and geometric methods for time-delay systems.

- M. Bronstein is a member of the ILC (Industrial Liaison Committee), advising body for the research center ORCCA (Ontario Research Center for Computer Algebra).
- M. Bronstein is a member of the editorial boards for the *Journal of Symbolic Computation* and for the *Algorithms and Computation in Mathematics* Springer monograph series.
- M. Bronstein was the program chair for the ISSAC'2003 conference (Philadelphia, August 2003).
- M. Bronstein has been elected vice-chair of the SIGSAM special interest group of ACM.
- J-A. Weil has organized the sessions "Equations differentielles I" and "Equations differentielles II" at the conference Journées Nationales de Calcul Formel (CIRM, January 2003).

9.2. Teaching

- Evelyne Hubert presented a practical course of computer algebra in the special week *Immersion Mathématique et Informatique* for second year students of the l'Ecole Nationale des Ponts et Chaussées. This was organized by Serge Piperno.
- Evelyne Hubert has taught computer science and computer algebra in the *classes préparatoires scientifiques* at the *Centre International de Valbonne* (72 hours).
- In the framework of the Graduiertenkolleg "Mathematics and Practice" of the University of Kaiserslautern (Germany), A. Quadrat was invited to lecture (9 hours) on effective algebraic analysis and its applications in October 2003. 20 graduate students, lecturers and professors, working either on effective algebra and symbolic computation or on control theory, attended the lectures.
- M. Bronstein participated as examiner on the doctoral panel of Anne Desidéri-Bracco (UNSA, December 2003).

9.3. Dissertations and internships

Doctorates in progress in the project:

1. Thomas Cluzeau, University of Limoges: *Algorithmique modulaire des équations différentielles linéaires*. Co-directed by Moulay Barkatou (University of Limoges) and J-A. Weil.
2. Nicolas Le Roux, University of Limoges: *Local study of nonlinear differential systems*. Co-directed by Moulay Barkatou (University of Limoges) and Evelyne Hubert.
3. Min Wu, UNSA and Academia Sinica (Beijing): *Factorization of systems of linear partial differential equations*. Co-directed by Ziming Li (Academia Sinica) and Manuel Bronstein.

Internships completed in 2003:

1. Damien Jamet, DEA internship from the University of Caen: "Intégration des fonctions algébriques", directed by Manuel Bronstein.
2. Simina Maris, DEA internship from the University of Limoges: "Une implémentation générique et efficace des bases involutives.", co-directed by Manuel Bronstein and Alban Quadrat.
3. Daniel Robertz, University of Aachen, "Computational Methods in Linear Control Theory", directed by Alban Quadrat.
4. Rohit Shrivastava, IIT Delhi, "Symbolic/Numeric genus computation", directed by Manuel Bronstein.

9.4. Conferences and workshops, invited conferences

M. Bronstein presented his work at the following conferences and workshops:

- the Journées Nationales de Calcul Formel (CIRM, January 2003).
- The MEGA'2003 conference (Kaiserslautern, May 2003).
- The first Joint Meeting of the Real Sociedad Mathematica Espanola (RSME) and American Mathematical Society (AMS) (Sevilla, June 2003).

In addition, he has attended the ISSAC'2003 conference (Philadelphia, August 2003) and the international conference on the occasion of the 60-th birthday of J-P. Ramis (Toulouse, September 2003), and gave a seminar presentation during an ORCCA joint lab meeting (UWO, December 2003).

S. Dalmas participated in the W3C technical plenary meeting (Boston, March 2003).

S. Dalmas and *M. Gaëtano* attended an OpenMath Thematic Network workshop (Bremen, November 2003).

M. Gaëtano attended the Mathematics on the Semantic Web MONET workshop (Eindhoven, May 2003).

E. Hubert

- was invited by Michael Singer to the Mathematical Science Research Institute in Berkeley (USA) in March and April.
- participated in March in the workshop *Computational Commutative Algebra* in MSRI.
- participated in the *CIMPA school on polynomial systems* in Buenos Aires (Argentina) organized by Alicia Dickenstein (University of Buenos Aires) and Ioannis Emiris (Galaad team, University of Athens) in July.
- presented her work on differential algebra for derivations that satisfy non trivial commutation rules at the *First Latin American workshop on polynomial systems* as well as at the computer algebra seminars of the *Laboratoire d'Arithmetique, de Calcul Formel et d'Optimisation* in Limoges and at the Laboratoire *Sciences et Technologies de l'Information et de la Communication à Polytechnique*.
- in collaboration with Nicolas Le Roux presented their work on the computation of power series solutions of nonlinear differential systems at the *International Symposium on Symbolic and Algebraic Computation* that was held in Philadelphia (USA) in August.
- visited twice the *Laboratoire d'Arithmetique, de Calcul Formel et d'Optimisation* in Limoges for collaboration with Nicolas Le Roux and Moulay Barkatou.

M. Przybylska presented her work at the following conferences and workshops:

- Young Researchers Marie Curie Meeting, Paris, France (March 2003).
- The first Joint Meeting of the Real Sociedad Mathematica Espanola (RSME) and American Mathematical Society (AMS) (Sevilla, June 2003).
- Workshop "Structural Dynamical Systems in Linear Algebra and Control. Computational Aspects", Capitolo-Monopoli, Bari, Italy (June 2003).
- XIV International Congress on Mathematical Physics, Lisbon, Portugal (July 2003).
- International Conference on the Occasion of the 60-th birthday of J-P. Ramis (Toulouse, September 2003).
- Geometry, Dynamical Systems and Celestial Mechanics. A tribute to Alain Chenciner, Paris, France (October 2003).

A. Quadrat presented his work at the following conferences and workshops:

- the Journées Nationales de Calcul Formel (CIRM, January 2003).
- The European Control Conference (Cambridge, UK, September 2003).
- A workshop of time-delay systems (INRIA Rocquencourt, September 2003).
- A summer school (Otzenhausen, Germany, September 2003).

He was also invited to represent the working group "Systèmes à Retards"of the *GdR Automatique* at the "Journées Nationales d'Automatique" (Valenciennes, June 2003). In the framework of our PAI Polonium (Sect. 8.2.3), *A. Quadrat* was invited for one week to the university of Zielona Góra in October 2003, where he gave seminar presentations at the the Institute of Engineering Cybernetics (Wroclaw) and the Institute of Control and Computational Engineering (Zielona Góra). He also gave a seminar presentation in the mathematics department of the University of Kaiserslautern (Germany) in October 2003.

J.-A. Weil has participated in:

- the MEGA'2003 conference (Kaiserslautern, May 2003).
- The first Joint Meeting of the Real Sociedad Mathematica Espanola (RSME) and American Mathematical Society (AMS) (Sevilla, June 2003).
- The ACA'2003 conference (Raleigh, July 2003).
- The ISSAC'2003 conference (Philadelphia, August 2003).

In addition, he has presented his work on differential Galois theory at the following conferences, workshops and seminars:

- the Journées Nationales de Calcul Formel (CIRM, January 2003).
- A Differential Galois Theory conference (Oberflockenbach, October 2003).

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