

Project-Team Café

Calcul Formel et Équations

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2. Overall Objectives

Our goal is to develop computer algebra methods and software for solving functional equations, i.e. equations where the unknowns represent functions rather than numerical values, as well as to foster the use of such methods in engineering by producing the programs and tools necessary to apply them to industrial problems. We study in particular linear and nonlinear differential and (q) -difference equations, partial and ordinary.

3. Scientific Foundations

3.1. Differential ideals and D-modules

Keywords: *D-modules, algebraic analysis, control theory, differential algebra, differential elimination, differential systems, formal integrability, formal integrability, holonomic systems, involution.*

Algorithms based on algebraic theories are developed to investigate the structure of the solution set of general differential systems. Different algebraic and geometric theories are the sources of our algorithms and making bridges between them is our challenge.

3.1.1. Formal integrability and differential elimination

Formal integrability is the first problem that our algorithms address. The idea is to *complete* a system of partial differential equation so as to be in a position to compute the *Hilbert differential dimension polynomial* or equivalently, its coefficients, the *Cartan characters*. Those provide an accurate measure of the arbitrariness that

comes in the solution set (how many arbitrary functions of so many variables). Closely related is the problem of determining the *initial conditions* that can be freely chosen for having a well-posed problem (i.e. that lead to existence and uniqueness of solutions). This is possible if we can compute all the differential relations up to a given order, meaning that we cannot obtain equations of lower order by combining the existing equations in the system. Such a system is called *formally integrable* and numerous algorithms for making systems of partial differential equations formally integrable have been developed using different approaches by E. Cartan [28], C. Riquier [66], M. Janet [40] and D. Spencer [72].

Differential elimination is the second problem that our algorithms deal with. One typically wants to determine what are the lowest differential equations that vanish on the solution set of a given differential system. The sense in which *lowest* has to be understood is to be specified. It can first be order-wise, as it is of use in the formal integrability problem. But one can also wish to find differential equations in a subset of the variables, allowing the model to be reduced.

The radical differential ideal generated by a set of differential polynomials \mathcal{S} , i.e. the left hand side of differential equations where the right hand side is zero, is the largest set of differential polynomials that vanish on the solution set of \mathcal{S} . This is the object that our algorithms manipulate and for which we compute adequate representations in order to answer the above questions.

In the nonlinear case the best we can hope for is to have information outside of some hypersurface. Actually, the radical differential ideal can be decomposed into components on which the answers to formal integrability and eliminations are different. For each component the *characteristic set* delivers the information about the singular hypersurface together with the quasi-generating set and membership test.

Triangulation-decomposition algorithms perform the task of computing a characteristic set for all the components of the radical differential ideal generated by a finite set of differential polynomials. References for those algorithms are the book chapters written by E. Hubert [5][4]. They are based on the differential algebra developed by Ritt [67] and Kolchin [42].

The objectives for future research in the branch of triangulation-decomposition is the improvement of the algorithms, the development of alternative approaches to certain class of differential systems and the study of the intrinsic complexity of differential systems.

Another problem, specific to the nonlinear case, is the understanding and algorithmic classification of the different behaviors of interference of the locus of one component on the locus of another. The problem becomes clear in the specific case of radical differential ideals generated by a single differential polynomial. One then wishes to understand the behavior of non singular solutions in the vicinity of singular solutions. Only the case of the first order differential polynomial equations is clear. Singular solutions are either the envelope or the limit case of the non singular solutions and the classification is algorithmic [67][39].

3.1.2. Algebraic analysis

When the set \mathcal{S} represent linear differential equations, we use the theory of D -modules (algebraic analysis, based on algebraic techniques such as module theory and homological algebra) as well as the formal integrability theories mentioned in the previous section (based on geometric techniques such as differential manifolds, jet spaces, involution and Lie groups of transformations). Using the duality between matrices of differential operators and differential modules, we can apply techniques that have been developed independently for those two approaches.

In addition, thanks to the works of B. Malgrange [50], V. Palamodov [55] and M. Kashiwara [41], the theory of D -modules yields new results and information about the algebraic and analytic properties of systems of linear partial differential equations, their solutions and associated geometric invariants (e.g. characteristic varieties). The above theories are becoming algorithmic thanks to the recent development of Gröbner bases [29] and involutive bases in rings of differential operators, enabling the implementation of efficient algorithms for making systems formally integrable, as well as for computing special closed-form solutions [53][54]. By using formal adjoints, it is now finally possible to algebraically study systems of linear partial differential equations, and our objectives in that field are: (i) to develop and implement efficient algorithms for computing

the polynomial and rational solutions of such systems and, further, for factoring and decomposing their associated D -modules; (ii) to study the links between the algebraic and analytic properties of such systems (since the algorithmic determination of the algebraic properties yields information about the analytic properties); (iii) to apply the above algorithms to the design and analysis of linear control systems.

3.2. Groups of transformations

Keywords: *Hamiltonian mechanics, closed-form solutions, differential Galois group, differential invariants, dynamical systems, formal integrability, linear systems of partial differential equations, nonlinear differential systems, symmetry, variational equations.*

3.2.1. Lie groups of transformations

Though not a major subject of expertise, the topic is at the crossroads of the algorithmic themes developed in the team.

The Lie group, or symmetry group, of a differential system is the (biggest) group of point transformations leaving the solution set invariant. Besides the group structure, a Lie group has the structure of a differentiable manifold. This double structure allows to concentrate on studying the tangent space at the origin, the Lie algebra. The Lie group and the Lie algebra thus capture the geometry of a differential system. This geometric knowledge is exploited to *solve* nonlinear differential systems.

The Lie algebra is described by the solution set of a system of linear partial differential equations, whose determination is algorithmic. The dimension of the solution space of that linear differential system is the dimension of the Lie group and can be determined by the tools described in Section 3.1. Explicit subalgebras can be determined thanks to the methods developed within the context of Section 3.2.2.

For a given group of transformations on a set of independent and dependent variables there exist invariant derivations and a finite set of differential invariants that generate all the differential invariants [73]. This forms an intrinsic frame for expressing any differential system invariant under this group action. This line of ideas took a pragmatic shape for computation with the general method of M. Fels and P. Olver [35] for computing the generating set of invariants. The differential algebra they consider has features that go beyond classical differential algebra. We are engaged in investigating the algebraic and algorithmic aspect of the subject.

3.2.2. Galois groups of linear functional equations

Differential Galois theory, developed first by Picard and Vessiot, then algebraically by Kolchin, associates a linear algebraic group to a linear ordinary differential equation or system. Many properties of its solutions, in particular the existence of closed-form solutions, are then equivalent to group-theoretic properties of the associated Galois group [70]. By developing algorithms that, given a differential equation, test such properties, Kovacic [43] and Singer [68] have made the existence of closed-form solutions decidable in the case of equations with polynomial coefficients. Furthermore, a generalization of differential Galois theory to linear ordinary difference equations [71] has yielded an algorithm for computing their closed-form solutions [36]. Those algorithms are however difficult to apply in practice (except for equations of order two) so many algorithmic improvements have been published in the past 20 years. Our objectives in this field are to improve the efficiency of the basic algorithms and to produce complete implementations, as well as to generalize them and their building blocks to linear partial differential and difference equations.

An exciting application of differential Galois theory to dynamical systems is the Morales-Ramis theory, which arose as a development of the Kovalevskaya-Painlevé analysis and Ziglin's integrability theory [75][76]. By connecting the existence of first integrals with branching of solutions as functions of complex time to a property of the differential Galois group of a variational equation, it yields an effective method of proving non-integrability and detecting possible integrability of dynamical systems. Consider the system of holomorphic differential equations

$$\dot{x} = v(x), \quad t \in \mathbb{C}, \quad x \in M, \quad (1)$$

defined on a complex n -dimensional manifold M . If $\phi(t)$ is a non-equilibrium solution of (1), then the maximal analytic continuation of $\phi(t)$ defines a Riemann surface Γ with t as a local coordinate. Together with (1) we consider its variational equations (VEs) restricted to $T_\Gamma M$, i.e.

$$\dot{\xi} = T(v)\xi, \quad \xi \in T_\Gamma M.$$

We can decrease the order of that system by considering the induced system on the normal bundle $N := T_\Gamma M / T\Gamma$ of Γ :

$$\dot{\eta} = \pi_{\star} (T(v)\pi^{-1}\eta), \quad \eta \in N$$

where $\pi : T_\Gamma M \rightarrow N$ is the projection. The system of $s = n - 1$ equations obtained in that way yields the so-called normal variational equations (NVEs). Their monodromy group $\mathcal{M} \subset \text{GL}(s, \mathbb{C})$ is the image of the fundamental group $\pi_1(\Gamma, t_0)$ of Γ obtained in the process of continuation of the local solutions defined in a neighborhood of t_0 along closed paths with base point t_0 . A non-constant rational function $f(z)$ of s variables $z = (z_1, \dots, z_s)$ is called an integral (or invariant) of the monodromy group if $f(g \cdot z) = f(z)$ for all $g \in \mathcal{M}$. In his two fundamental papers [75][76], Ziglin showed that if (1) possesses a meromorphic first integral, then \mathcal{M} has a rational first integral. Ziglin found a necessary condition for the existence of a maximal number of first integrals (without involutivity property) for analytic Hamiltonian systems, when $n = 2m$, in the language of the monodromy group. Namely, let us assume that there exists a non-resonant element $g \in \mathcal{M}$. If the Hamiltonian system with m degrees of freedom has m meromorphic first integrals $F_1 = H, \dots, F_m$, which are functionally independent in a connected neighborhood of Γ , then any other monodromy matrix $g' \in \mathcal{M}$ transforms eigenvectors of g to its eigenvectors.

There is a problem however in making that theory algorithmic: the monodromy group is known only for a few differential equations. To overcome that problem, Morales-Ruiz and Ramis recently generalized Ziglin's approach by replacing the monodromy group \mathcal{M} by the differential Galois group \mathcal{G} of the NVEs. They formulated [51] a new criterion of non-integrability for Hamiltonian systems in terms of the properties of the connected identity component of \mathcal{G} : if a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighborhood of the analytic curve Γ , then the identity component of the differential Galois group of NVEs associated with Γ is Abelian. Since \mathcal{G} always contains \mathcal{M} , the Morales-Ramis non-integrability theorem always yields stronger necessary conditions than the Ziglin criterion.

When applying the Morales-Ramis criterion, our first step is to find a non-equilibrium particular solution, which often lies on an invariant submanifold. Next, we calculate the corresponding VEs and NVEs. If we know that our Hamiltonian system possesses k first integrals in involution, then we can consider VEs on one of their common levels, and the order of NVEs is equal to $s = 2(m - k)$ [23][24]. In the last step we have to check if the identity component of \mathcal{G} is Abelian, a task where the tools of algorithmic Galois theory (such as the Kovacic algorithm) become useful. In practice, we often check only whether that component is solvable (which is equivalent to check whether the NVEs have Liouvillian solutions), because the system is not integrable when that component is not solvable.

Our main objectives in that field are: (i) to apply the Morales-Ramis theory to various dynamical systems occurring in mechanics and astronomy; (ii) to develop algorithms that carry out effectively all the steps of that theory; (iii) to extend it by making use of non-homogeneous variational equations; (iv) to generalize it to various non-Hamiltonian systems, e.g. for systems with certain tensor invariants; (v) to formulate theorems about partial integrability of dynamical systems and about real integrability (for real dynamical systems) in the framework of the Morales-Ramis theory;

3.3. Mathematical web services

Keywords: *Computer algebra, MathML, OpenMath, Web services, communication, deductive databases, formula databases.*

The general theme of this aspect of our work is to develop tools that make it possible to share mathematical knowledge or algorithms between different software systems running at arbitrary locations on the web.

Most computer algebra systems deal with a lot of non algorithmic knowledge, represented directly in their source code. Typical examples are the values of particular integrals or sums. A very natural idea is to group this knowledge into a database. Unfortunately, common database systems are not capable to support the kind of mathematical manipulations that are needed for an efficient retrieval (doing pattern-matching, taking into account commutativity, etc.). The design and implementation of a suitable database raise some interesting problems at the frontier of computer algebra. We are currently developing a prototype for such a database that is capable of doing some deductions. Part of our prototype could be applied to the general problem of searching through mathematical texts, a problem that we plan to address in the near future.

The computer algebra community recognized more than ten years ago that in order to share knowledge such as the above database on the web, it was first necessary to develop a standard for communicating mathematical objects (via interprocess communication, e-mail, archiving in databases). We actively participated in the definition of such a standard, OpenMath (partly in the course of a European project). We were also involved in the definition of MathML by the World Wide Web Consortium. The availability of these two standards are the first step needed to develop rich mathematical services and new architectures for computer algebra and scientific computation in general enabling a transparent and dynamic access to mathematical components. We are now working towards this goal by experimenting with our mathematical software and emerging technologies (Web Services) and participating in the further development of OpenMath.

4. Application Domains

4.1. Panorama

We have applied our algorithms and programs for computing differential Galois groups to determine necessary conditions for integrability in mechanical modeling and astronomy (see 6.1). We also apply our research on partial linear differential equations to control theory, for example for parameterization and stabilization of linear control systems (see 6.6 and 6.7). Other applications include biology, where our algorithms for solving recurrence equations have been applied to the steady-state equations of nonhomogeneous Markov chains modelling the evolution of microsatellites in genomes.

5. Software

5.1. Maple package *diffalg*

Keywords: *analysis of singular solutions, differential algebra, differential elimination, nonlinear differential systems, triangulation-decomposition algorithms.*

Participant: Evelyne Hubert.

The *diffalg* library has been part of the commercial release of Maple since Maple V.5 (with an initial version of F. Boulier) and has evolved up to Maple 7. The library implements a triangulation-decomposition algorithm for polynomially nonlinear systems and tools for the analysis of singular solutions.

A new version of the library is now available. Its high point is the implementation of algorithms for differential polynomial rings where derivations satisfy nontrivial commutation rules [38]. This new version also incorporates a specific treatment of parameters as well as improved algorithms for higher degree polynomials [18].

5.2. Library *OreModules* of *Mgfun*

Keywords: *Effective algebraic analysis, Gröbner basis, Ore algebras, linear systems.*

Participants: Frédéric Chyzak, Alban Quadrat [correspondent], Daniel Robertz.

The library OREMODULES of MGFUN has been developed on some effective aspects of algebraic analysis. It is dedicated to the study of under-determined linear systems over some Ore algebras (e.g. ordinary differential equations, partial differential equations, differential time-delay equations, discrete equations) and their applications in mathematical physics (e.g. research of potentials, computations of the field equations and the conservation laws) and in linear control theory (e.g. controllability, observability, parameterizability, flatness of multidimensional systems with varying coefficients). The main novelty of OREMODULES is to use the recent development of the Gröbner basis over some Ore algebras (non-commutative polynomial rings) in order to effectively check some properties of module theory (e.g. torsion/torsion-free/reflexive/projective/stably free/free modules) and homological algebra (e.g. free resolutions, split exact sequences, duality, extension functor, dimensions).

A library of examples (e.g. a two pendulum mounted on a car, time-varying systems, systems of algebraic equations, a wind tunnel model, a two reflector antenna, an electric transmission line, Einstein equations, Maxwell equations, linear elasticity, Lie-Poisson structures) has been developed in order to illustrate the main functions of OREMODULES.

The second release of OREMODULES, developed in 2004, is freely available together with the library of examples.

5.3. Library libaldor

Keywords: *Aldor, arithmetic, data structures, standard.*

Participant: Manuel Bronstein.

The LIBALDOR library, under development in the project for several years, became the standard ALDOR library, bundled with the compiler distribution since 2001. Version 1.0.2 of LIBALDOR was released in May 2004, together with version 1.0.2 of the ALDOR compiler.

5.4. Library Algebra

Keywords: *commutative algebra, computer algebra, linear algebra, polynomials.*

Participants: Manuel Bronstein [correspondent], Marc Moreno-Maza.

The ALGEBRA library, written in collaboration with Marc Moreno-Maza (UWO, London, Ontario), is now also bundled with the ALDOR compiler, starting with version 1.0.2, which has been released in May 2004. It is intended to become a standard core for computer algebra applications written in ALDOR.

5.5. Library Σ^{it}

Keywords: *computer algebra, difference equations, differential equations, systems.*

Participant: Manuel Bronstein.

The Σ^{it} library contains our algorithms for solving functional equations. Following the 2003 implementation of an algorithm [77] for computing invariants of differential Galois groups, we have implemented in 2004 a complete stand-alone solver for linear ordinary differential equations of order 3, which is part of version 1.0.2 of Σ^{it} , released in December 2004. That version also contains a new algorithm for computing the eigen-rings of scalar linear ordinary differential operators, as well as several modular algorithms (rational solutions and eigenrings of differential operators and systems) from the thesis of T. Cluzeau [30]. Our stand-alone solver now powers the the web service at http://www-sop.inria.fr/cafe/Manuel.Bronstein/sumit/bernina_demo.html.

6. New Results

6.1. Integrability analysis of dynamical systems

Keywords: *Darboux polynomials, Darboux points, Hamiltonian systems, chaos, homogeneous Hamiltonian systems, integrability.*

Participant: Maria Przybylska.

In cooperation with Jacques-Arthur Weil (Université de Limoges) and with A. Maciejewski (University of Zielona Góra, Poland) we studied in detail the Morales-Ramis theory from point of view its applications to integrability analysis. Our investigations show that the technique of symmetric powers and local monodromy are very useful for checking of the abelianity for differential Galois groups. Additionally, as by-product we checked integrability of various systems depending on parameters important in mechanics [46][48][49][10][13][12][11].

However the most interesting results concerns the class of natural Hamiltonian systems with two degrees of freedom $H = \frac{1}{2} \sum_{i=1}^n p_i^2 + V(q_1, \dots, q_n)$ where V is a homogeneous polynomial. In [8] we gave the necessary and sufficient conditions for meromorphic integrability of such systems in the case when $n = 2$ and $\deg V = 3$. To obtain this result it was necessary to apply an extension of the Morales-Ramis theory to higher order variational equations. However, in the proof of this result the crucial role is played by the relation between the non-trivial eigenvalues λ_i of the Hessian of the potential calculated at different Darboux points

$$\lambda_1 \lambda_2 \lambda_3 - (\lambda_1 + \lambda_2 + \lambda_3) + 2 = 0.$$

A Darboux point is a non-zero solution of equation $\nabla V(q) = q$, $q \in \mathbb{C}^n$. For a generic potential of degree 3 when $n = 2$ there are 3 Darboux points. One eigenvalue of Hessian $\nabla^2 V(q)$ calculated at i -th Darboux point is equal to $\deg V - 1 = 2$, the remaining one is denoted by λ_i .

It appeared that the existence of such a relation is a general property of Hamiltonian system with two degrees of freedom and homogeneous potential of an arbitrary degree k . If we define $\Lambda_i = \lambda_i - 1$, then this general relation takes the form $\sum_{i=1}^p \Lambda_i^{-1} = -1$, where p denotes the number of Darboux points and $0 < p \leq k$. This relation can be obtained by the residue theorem for a certain meromorphic differential form depending on V [47]. We also showed that the existence of this relation enables to find all meromorphically integrable homogeneous potentials of degree four [47].

In [9] we also proved by means of Darboux polynomials that for natural Hamiltonian systems $H = \frac{1}{2} \sum_{i=1}^n \mu_i p_i^2 + V(q)$, where $\mu_i \in \mathbb{C}$ and $V(q) \in \mathbb{C}[q_1, \dots, q_n]$ the existence of rational first integral implies immediately the existence of polynomial first integral. Furthermore, in a case when the degree of polynomial is odd, its only Darboux polynomials are first integrals.

Our studies of the class of Gross-Neveu systems [49] showed that it is sometimes easier to observe a chaos in a Hamiltonian system by analyzing a Poincaré section not directly in the original system but in a conjugated one obtained by a canonical change of variables: $q_k \rightarrow iq_k$ and $p_k \rightarrow -ip_k$. This phenomenon is related with the fact that the sets $V(q)$ and $V(iq)$ of admissible values of energy for which Poincaré sections are made, are different.

6.2. Algorithms for nonlinear differential systems

Participant: Evelyne Hubert.

We introduced new ideas to improve the efficiency and rationality of a triangulation decomposition algorithm (cf. Section 3.1). On the one hand we wished to identify and isolate the polynomial remainder sequences in the triangulation-decomposition algorithm. Subresultant polynomial remainder sequences are then used to compute them and their specialization properties are applied for the splittings. The gain would be twofold: control of expression swell and reduction of the number of splittings. On the other hand, we wished to remove the role that initials had in previous triangulation-decomposition algorithms. They are not needed in theoretical results and it was expected that they need not appear in the input and output of the algorithms.

This has been achieved for systems of ordinary differential equations. The article [18] was accepted at ISSAC'04. It also contains new algorithms to compute a subsequent characteristic decomposition from the output of the triangulation decomposition algorithm where the initials need not appear.

6.3. Algebraic structures for differential invariants

Participant: Evelyne Hubert.

We have carried on our exploration of the algebraic aspects of differential invariants and the method of the moving frame [35]. A special emphasis was put on making precise the syzygy structure of the differential invariants computed by the moving frame.

The generalisation of differential algebra to non commuting derivations [37] that was developed in that context found some other sources of applications. In particular the treatment of the equivalence problem in the line of Cartan's original method, that was turned recently into an implemented algorithm [52], will benefit of the theory and related software (see Section 5.1).

We see here the emergence of computer algebra in differential geometry. Further research and collaborations in that direction were discussed during visits at the University of Minneapolis and the University of North Carolina, as well as at the CMS conference in Halifax.

6.4. Algorithms for solving linear ordinary equations and systems

Keywords: *algebraic curves, algebraic functions, integration, linear recurrent sequences.*

Participants: Manuel Bronstein, Adrien Poteaux.

6.4.1. Symbolic Integration:

Following Jamet's work in 2003 on the lazy Hermite integration of algebraic functions, we have investigated the possibility of lazily computing the logarithmic part of the integral of an algebraic function. A prototype lazy integrator that avoids desingularizing the curve when the integral can be expressed with one logarithmic term has been implemented in MAPLE by Adrien Poteaux. Such integrals occur when solving linear ordinary differential equations with finite Galois groups. In addition, we have investigated generalizations of the Risch–Norman integration procedure [33][34][32] to non-elementary integrands, such as Lambert's W function, and have discovered a refinement of Liouville's Theorem on integration that predicts the possible new logarithms that can appear in an integral [27].

6.4.2. Linear ordinary difference operators:

Constructions used in solving linear ordinary difference equations have been generalized to coefficient fields with characteristic $p > 0$, and applied to the study of linear recurrent sequences, obtaining a new proof that over finite fields, sequences generated by linear recurrences with polynomial coefficients are periodic [7].

6.5. Algorithms for solving partial differential equations

Keywords: *D-modules, Partial differential equations, Picard–Vessiot extensions, decomposition, factorization, partial difference equations, polynomial solutions, rational solutions.*

Participants: Manuel Bronstein, Min Wu.

6.5.1. Factorization of linear systems of partial functional equations:

We continued our work towards developing effective algorithms and programs to factor systems of linear partial differential or (q)-difference equations with finite-dimensional solution spaces (called fLPDEs). The Li-Schwarz-Tsarev algorithm [45] is a generalization of Beke's factorization algorithm [25] from linear ODEs to fLPDEs. That generalization uses Beke spaces and relies on the Li-Schwarz algorithm [44] to compute rational solutions of Riccati-like partial differential equations. We are investigating instead the D -module interpretation of the Beke algorithm [70], which allows us to express not only Beke's algorithm, but also the Plücker relations and the eigenring decomposition method [69].

Using the module of formal solutions over a ring of multivariate Laurent–Ore polynomials, we have shown this year the existence of Picard–Vessiot extensions (hence of Galois groups) for arbitrary fLPDEs. This work is being submitted to the ISSAC'2005 conference.

6.6. Linear systems over Ore algebras and applications

Keywords: *(under-/over-determined) linear systems, Algebraic analysis, Gröbner basis for non-commutative polynomial rings, Ore algebras, multidimensional systems.*

Participants: Alban Quadrat, Daniel Robertz.

In [21][17], we study some structural properties of linear systems over Ore algebras (e.g. systems of ordinary differential equations, systems of differential time-delay equations, systems of partial differential equations, systems of difference equations). Using the recent development of Gröbner basis over some Ore algebras (i.e. non-commutative polynomial rings), we show how to make effective some important concepts of homological algebra (e.g. free resolutions, split exact sequences, duality, extension functor, dimensions). Then, using these results, we obtain some effective algorithms which check the different structural properties of under-determined linear systems over Ore algebras, developed using module theory (e.g. torsion/torsion-free/reflexive/projective/stably free/free modules). In particular, we explain why these properties are related to the possibility to successively parameterize all solutions of a system and its parameterizations. Moreover, we show how these properties generalize the well-known concepts of primeness, developed in the literature of multidimensional systems, to systems with varying coefficients. Then, using a dictionary between the structural properties of under-determined linear systems over Ore algebras and some concepts of linear control theory, we show that the previous algorithms allow us to effectively check whether or not a linear control system over certain Ore algebras is (weakly, strongly) controllable, observable, parameterizable, flat, π -free... Finally, using the implementation of all these algorithms in OREMODULES (see 5.2), these results are illustrated on different systems (e.g. two pendulum mounted on a car, time-varying systems, systems of algebraic equations, a wind tunnel model, a two reflector antenna, an electric transmission line, Einstein equations, Maxwell equations, linear elasticity, Lie-Poisson structures). Let us remark that the problem of parameterizing all the solutions of linear controllable multidimensional systems has been extensively studied by the school of M. Fliess in France and J. C. Willems in the Netherlands [56]. Hence, our results allow us to effectively answer that problem for a large classes of systems considered in the literature.

For linear systems of partial differential equations, we have shown [14] how to use the previous results in order to study some variational problems. In particular, we show how the Lagrangian given by the variational problem links the sequence formed by the differential operator describing the system and its parameterization and the sequence formed by their formal adjoints (Poincaré duality). Using this result, we give different new equivalent forms for the optimal system (with or without the Lagrangian multipliers). These results are illustrated on examples coming from linear elasticity, electromagnetism and linear quadratic optimal problems.

We have recently obtained [65] a necessary and sufficient conditions so that a general differential module is isomorphic to the direct sum of its torsion submodule and its torsion-free part. These new results allowed us to parametrize all solutions of a linear system of partial differential equations by glueing the autonomous elements of the system to the parametrization of its controllable sub-behaviour. These results can be extended in order to check whether a general module, defined by means of a system of equations over an Ore algebra, is a direct summand of a given module.

Finally, we have written an introduction to algebraic analysis [62] (algebraic D -modules) and a survey on effective methods for differential time-delay systems [63].

6.7. Algebraic analysis approach to infinite-dimensional linear systems

Keywords: *Systems of partial differential equations or differential time-delay equations, algorithms, internal/strong/simultaneous/robust/optimal stabilization, module theory.*

Participant: Alban Quadrat.

We have studied [59][60] *infinite-dimensional linear systems* [31], namely some classes of systems of partial differential equations (e.g. wave, heat, Euler-Bernoulli equations), differential time-delay systems (e.g. transport equation, transmission lines), fractional differential systems... usually encountered in mathematical physics, within an algebraic analysis framework. Let us remark that until today, no one has been able to

show how to incorporate the initial/boundary conditions of the system in the classical differential module (D -module) approach. In order to face this problem, using symbolic calculus (i.e. Laplace transform and symbolic integration), we transform an invariant infinite-dimensional linear system into a transfer matrix, i.e. into functional relations between the inputs and the outputs of the system. Then, using *the fractional representation approach to systems* [74], we show [59][60] how to develop a module-theoretic approach to infinite-dimensional linear systems. An important issue in control theory is to stabilize an unstable system (e.g. a system with an infinite number of unstable poles) by adding a controller in feedback to the system (internal stabilization). Using the module-theoretic approach to infinite-dimensional linear systems, we have obtained [59][60] general necessary and sufficient conditions for the existence of (weakly) left/right/doubly coprime factorizations and for internal stabilizability. To our knowledge, these necessary and sufficient conditions are the most general ones in the literature and give an answer to a question of M. Vidyasagar, B. Francis and H. Schneider about the links between internal stabilizability and the existence of doubly coprime factorizations [74]. Finally, we characterize all the algebras such that one of the previous properties is always satisfied and we develop some algorithms in order to check these different properties.

We have also shown in [16][19] how the algebraic concept of *stable range*, introduced in algebra by H. Bass, plays a central role in the *strong stabilization problem* (stabilization of a plant by means of a stable controller) [74]. In particular, we exhibit the particular form of certain stabilizing controllers in which the size of the unstable part depends on the stable range of the system. This result allows us to prove that any multi-input multi-output stabilizable system over the rings $H_\infty(\mathbb{D})$, $H_\infty(\mathbb{C}_+)$, W_+ and $A(\mathbb{D})$ is strongly stabilizable. Let us remark that this result answers a question asked in [26].

We have also shown [57] how the operator-theoretic approach to linear systems [31] is dual to the module-theoretic approach [58]. This new theory plays a similar role as the behavioral approach to multidimensional linear systems developed by J. C. Willems and his school [56]. In particular, using the algebraic concept of fractional ideals, we exhibit the precise domain and graph of an internal stabilizable system with a single-input and single-output. This result generalizes all the ones known in the literature.

Finally, in a couple of submitted papers [64][61], we show, using the algebraic concepts of lattices, how to generalize the well-known *Youla-Kučera parameterization of all stabilizing controllers* [74] for multi-input multi-output systems which do not necessarily admit coprime factorizations. This new parameterization allows us to rewrite the problem of finding the optimal stabilizing controllers (for a certain norm) as affine, and thus, convex problems. In particular, we solve the well-known conjecture of Z. Lin on the equivalence between internal stabilizability and the existence of doubly coprime factorizations for multidimensional systems [64] [20].

All these results have been summarized in a recently appeared survey [15].

7. Contracts and Grants with Industry

7.1. Waterloo Maple Inc.

Participant: Manuel Bronstein.

Since 2003, WMI (makers of the computer algebra system MAPLE) supports the collaboration between CAFÉ and Prof. Abramov's group at the Russian academy of science, by paying for travel and living expenses. In exchange, WMI gets early reports on the results of that collaboration as well as a faster track between algorithmic developments and their distribution as part of MAPLE. A prolongation for 2005 is awaiting signature.

8. Other Grants and Activities

8.1. National initiatives

A. Quadrat is a member of the working group "Systèmes à Retards" of the *GdR Automatique*.

8.2. European initiatives

8.2.1. *OpenMath*

Participants: Stéphane Dalmas, Marc Gaëtano.

CAFÉ was a participant in the *OpenMath* (IST-2000-28719) Thematic Network that ended in June 2004. This network was a follow on from the earlier ESPRIT Project. The network's main activities was to organise workshops bringing together people working on OpenMath from around the world, to provide a continued focus-point for the development of the OpenMath Standard, to facilitate European participation in the W3C Math Working group, to coordinate the development of OpenMath and MathML tools, to coordinate the development of OpenMath and MathML applications and to disseminate information about OpenMath and MathML. The activity of the network culminated with the release **version 2.0 of the OpenMath standard** in July 2004.

The membership of the network was NAG Ltd (UK, coordinator), the University of Bath (UK), Stilo Technology Ltd (UK), INRIA, the University of St Andrews (UK), the Technical University of Eindhoven (Netherlands), Springer Verlag (Germany), the University of Nice Sophia Antipolis, ZIB (Germany), Exploit Research (Italy), RISC (Austria), German Research Centre for Artificial Intelligence (Germany), the University of Helsinki (Finland).

8.2.2. *PAI Polonium*

K. Avratchenkov (MISTRAL), P.A. Bliman (SOSSO) and A. Quadrat (CAFÉ) have a collaboration with Prof. K. Galkowski's group at the University of Zielona Góra (Poland) within the framework of an exchange research program PAI *Polonium* entitled "Theory and applications of n -dimensional systems, delay systems and iterative learning control" started in 2003 and continued in 2004.

8.2.3. *ECO-NET Program*

Following the termination of our Liapunov projects with Prof. S.A. Abramov (Moscow), our collaboration has been extended to include Prof. M. Petkovšek (Ljubljana). This collaboration was supported in 2004 by the **ECO-NET** program under grant number 08119TG. During 2004, we extended our algorithm for computing rational solutions of differential systems [22] in order to compute all the regular solutions at a singularity. We also continued our progress towards a complete algorithm for factoring linear ordinary difference operators with coefficients in nested towers of hypergeometric extensions. An extension to 2005 of this collaboration has been submitted to ECO-NET.

8.2.4. *European training site*

Under the supervision of A. Quadrat, the project "Computational methods in linear control systems" of Daniel Robertz, PhD student at the University of Aachen (Germany), has been granted from the **Control Training Site (CTS)**.

8.3. Other international initiatives

8.3.1. *China*

Following the termination of our PRA with Z. Li (Academia Sinica), our collaboration continues with the co-direction of the thesis of Min Wu, who is alternating 6-month stays in our project and in Beijing.

8.4. International networks and working groups

Stéphane Dalmas is a member of the Math Interest Group of the World Wide Web Consortium. This group is responsible for maintaining existing documents related to MathML, working with other W3C groups and provide general support on MathML and mathematics on the Web. Stéphane Dalmas was a member of the W3C Math Working Group, the group responsible for defining MathML, an XML application for describing mathematical notation and capturing both its structure and content. The goal of MathML is to

enable mathematics to be served, received, and processed on the World Wide Web, just as HTML has enabled this functionality for text.

8.5. Visiting scientists

8.5.1. Europe

Under the sponsorship of the Control Training Site (Sect. 8.2.4), Daniel Robertz visited our project for two months (February–March) in order to collaborate on the *OreModules* library (see 5.2).

Within the framework of our ECO-NET project (Sect. 8.2.3), S.A. Abramov (CC RAS, Moscow), D.E. Khmel'nov (CC RAS, Moscow), M. Petkovšek (University of Ljubljana) and H. Zakrajšek (University of Ljubljana) visited our project for two weeks in June. In addition, S.A. Abramov came for a one-week visit in December.

Ralf Hemmecke (RISC Linz, Austria) visited our project for one week in May in order to collaborate on the design of our LIBALDOR and ALGEBRA libraries (see 5.3) to include non-commutative polynomials. A PAI Amadeus 2005 for this collaboration has been approved.

Wilfrid Kendall (University of Warwick) visited our project for one week in December in order to investigate the applicability of differential algebra to stochastic calculus.

Andrzej Maciejewski (Zielona Góra, Poland) and Jacques-Arthur Weil (Limoges) visited our project together for one week in December, to work with M. Przybylska on the integrability of dynamical systems.

8.5.2. Outside Europe

As part of our joint library development for the ALDOR compiler (see 5.3), Stephen Watt (UWO, Canada) and Marc Moreno Maza (UWO, Canada) visited our project for one week in May. A 2005 proposal to the France–Canada research fund has been submitted for this collaboration.

Irina Kogan (NCSU, USA) visited our project for one week in December to work with E. Hubert on the algebraic aspect of the moving frame construction and its application.

Greg Reid (UWO, Canada) spent a month out of his sabbatical with our group.

Shiva Shankar (Chennai Mathematical Institute) visited our project for one month in November–December 2004 to work with A. Quadrat on developing a behaviour approach to multidimensional systems described by linear partial differential equations with polynomial and rational coefficients.

9. Dissemination

9.1. Leadership within scientific community

- Evelyne Hubert was a member of the program committee of the **International Conference for Polynomial System Solving** held in Paris in November 2004.
- Evelyne Hubert is member of the council of the association *femmes & mathématiques*. As such she participates to the monthly meetings of the council to propose, discuss and undertake the actions of the association. Those aim at promoting scientific education and careers for women.
- Evelyne Hubert chaired the organization of the *forum des jeunes mathématiciennes* held in Paris in January 2004. The forum is a francophone conference organized every two years by the association *femmes & mathématiques*. It is an occasion for senior and junior female mathematicians to meet so as to favor mentoring, role model and identification. Beside the scientific talks, the forum stages debates on subjects relating to science and education and talks in humanities.
The focus of the forum this year was on Mathematics, Computer Science and Life sciences. It received the financial support of both INRA and INRIA. The conference was a fair success: it staged 25 talk and gathered around 60 participants.

- Evelyne Hubert organizes the *Séminaires Croisés*, in collaboration with the service REV. The *séminaires croisés* is a rather unique initiative of global scientific animation within the site of INRIA Sophia Antipolis. In 2004, there were 7 thematic days involving 25 project teams with a total of 25 talks. A renewed success.
- Evelyne Hubert ended in March her 3 year mandate as a member of the Committee of Doctoral Studies of INRIA Sophia (chaired by Thierry Vieville).
- Evelyne Hubert is a member of the **COLORS** committee chaired by Rose Dieng.
- M. Bronstein is a member of the editorial boards for the *Journal of Symbolic Computation* and for the *Algorithms and Computation in Mathematics* Springer monograph series.
- M. Bronstein is vice-chair of the **SIGSAM** special interest group of ACM since August 2003.

9.2. Teaching

- Evelyne Hubert presented a practical course of computer algebra in the special week *Immersion Mathématique et Informatique* for second year students of the l'Ecole Nationale des Ponts et Chaussées. This was organized by Serge Piperno.
- Evelyne Hubert taught computer science and computer algebra in the *classes préparatoires scientifiques* at the *Centre International de Valbonne*.
- M. Bronstein participated as examiner on the doctoral panels of Thomas Cluzeau (Limoges, September 2004) and Guillaume Chèze (UNSA, December 2004), and as reviewer on the doctoral panel of Philippe Gaillard (Rennes, October 2004).

9.3. Dissertations and internships

Doctorates completed in 2004:

1. Thomas Cluzeau, University of Limoges: *Algorithmique modulaire des équations différentielles linéaires*. Co-directed by Moulay Barkatou (University of Limoges) and J-A. Weil (Member of the CAFÉ project during 2002–2003).

Doctorates in progress in the project:

1. Nicolas Le Roux, University of Limoges: *Local study of systems of partial differential equations*. Co-directed by Moulay Barkatou (University of Limoges) and Evelyne Hubert.
2. Min Wu, UNSA and Academia Sinica (Beijing): *Factorization of systems of linear partial functional equations*. Co-directed by Ziming Li (Academia Sinica) and Manuel Bronstein.

Internships completed in 2004:

1. Adrien Poteaux, DEA internship from the University of Limoges: *Intégration des fonctions algébriques : partie logarithmique*, directed by Manuel Bronstein.
2. Daniel Robertz, University of Aachen, *Computational Methods in Linear Control Theory*, directed by Alban Quadrat.

9.4. Conferences and workshops, invited conferences

M. Bronstein presented his work on parallel integration at the computer algebra seminar of the STIX team (Ecole Polytechnique), and gave an invited conference in the cycle *Sensibilisation et formation aux outils de calculs, aux ressources numériques en ligne et aux nouvelles technologies* at the University of Montpellier. He also attended the **ISSAC'2004** conference (Santander, Spain, July 2004) and an ECO-NET workshop in Slovenia (Bled, November 2004).

M. Gaëtano attended the *10 Years of OpenMath* workshop (Helsinki, Finland, May 2004).

E. Hubert

- participated to the *Forum des Jeunes Mathématiciennes; Mathématiques, Informatique et Sciences du Vivant* held in Paris in January.
- visited the *Laboratoire d'Arithmétique, de Calcul Formel et d'Optimisation* in Limoges for collaboration with Nicolas Le Roux and Moulay Barkatou. She gave there a talk at the computer algebra seminar.
- visited P. Olver at the *University of Minneapolis* (Minnesota) and presented her work at the math-physics seminar.
- visited North Carolina State University in Raleigh and gave a talk at the *Workshop on Differential Algebra and Symbolic Computation* after the invitation of A. Szanto.
- participated to the **Canadian Mathematics Society Summer Meeting** in Halifax (Nova Scotia). She presented her work at the session *Applications of Invariant Theory to Differential Geometry* organized by R. Milson and M. Fels.
- was invited to present *difalg* (see Section 5.1) in a tutorial talk at the AARMS (Atlantic Association for Research in the Mathematical Sciences) workshop on Symbolic Computation. The workshop was organized by R. Milson, M. Fels, and A. Coley. It was held as a satellite to the CMS meeting in Halifax (Nova Scotia) in June.
- gave a talk at the *Encuentro de Álgebra Computacional y Aplicaciones* held in Santander (Spain) in June.
- presented her accepted paper at the *International Symposium on Symbolic and Algebraic Computation* held in Santander (Spain) in July.
- was a colloquium speaker at the **Research Institute for Symbolic Computation** in Linz (Austria). She was invited there for a week by R. Hemmecke and F. Winkler and gave a second talk at the computer algebra seminar.
- took part to the **International Conference for Polynomial System Solving** held in Paris in November 2004 in the honor of D. Lazard retirement.

M. Przybylska presented results her works during the following conferences and workshops:

- *International Conference on Asymptotic Theories and Painlevé Equations* (Angers, France, June 2004).
- *4th Conference on Poisson Geometry* (Luxembourg, June 2004).
- *First Joint Canada-France meeting of the mathematical sciences* (Toulouse, France, July 2004).
- *Fifth International Symposium on Classical and Celestial Mechanics* (Velikie Luki, Russia, August 2004).

A. Quadrat presented his work at the **MTNS'2004** conference (Louvain, Belgium, July 2004) and at the **2nd CNRS-NSF workshop on applications of time-delay systems** (Nantes, France, September 2004).

M. Wu attended the **ISSAC'2004** conference (Santander, Spain, July 2004) and an ECO-NET workshop in Slovenia (Bled, November 2004).

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