



INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Team salsa

*Solvers for ALgebraic Systems and
Applications*

Rocquencourt

THEME SYM

Activity
R
Report

2004

Table of contents

1. Team	1
2. Overall Objectives	1
3. Scientific Foundations	1
3.1. Introduction	1
3.2. Gröbner basis and triangular sets	3
3.3. Zero-dimensional systems	6
3.4. Positive-dimensional and parametric systems	7
3.4.1. Critical point methods	7
3.4.2. Parametric systems	9
4. Application Domains	11
4.1. Panorama	11
4.2. Cryptography	11
4.3. Parallel robots	12
4.4. Serial robots	14
4.5. Signal Processing	15
5. Software	15
5.1. GC	15
5.2. UDX	16
5.3. MPFI	16
5.4. MPAI	16
5.5. Interfaces	16
5.6. Gb	16
5.7. FGb Servers	17
5.8. RS	17
5.9. Triangular Decompositions	17
5.10. RAGlib	17
5.11. TCI	18
5.12. TSPR	18
5.13. NetTask	18
6. New Results	19
6.1. Interval Arithmetic	19
6.2. Arithmetic of $C_{3,4}$ curves	19
6.3. Univariate polynomials	19
6.4. Generalized Normal Forms	19
6.5. Complexity of Gröbner basis	19
6.6. Gröbner bases over finite field	20
6.7. Zero-dimensional Systems	20
6.8. Parametric Polynomial Systems	20
6.9. Parametric varieties and rational maps	21
6.10. Positive dimensional systems of equations	21
6.11. Generalized critical values and semi-algebraic systems	21
7. Contracts and Grants with Industry	22
7.1. CELAR (DGA)	22
7.2. Sciface Software	22
7.3. Thales	22
7.4. National Actions	22
7.4.1. Ministry Grant (ACI) "Cryptologie"	22

7.5. European Actions	23
7.5.1. European Network RAAG	23
8. Dissemination	23
8.1. Scientific Animation	23
8.1.1. ICPSS	23
8.1.2. Invited lectures	23
9. Bibliography	24

1. Team

Head of project-team

Fabrice Rouillier [research scientist - INRIA]

Vice-head of project team

Jean-Charles Faugère [research scientist CNRS]

administrative assistant

Laurence Bourcier [INRIA, part time]

Staff member Université Pierre et Marie Curie

Philippe Aubry [assistant professor]

Daniel Lazard [professor]

Mohab Safey El Din [assistant professor]

Philippe Trébuchet [teaching assistant]

Ph. D. student

Gwenolé Ars [DGA, defense Dec. 2004/ Jan. 2005 - J.C. Faugère]

Magali Bardet [teaching assistant, defended 2004/12/08 - J.C. Faugère]

Solen Corvez [BDI CNRS, defense planned in 2005 - F. Rouillier]

Amir Hashemi [Sfere grant, defense planned in 2005 - D. Lazard]

Sylvain Lacharte [CIFRE grant with Thalès, defense planned in 2006 - J.C. Faugère]

Guillaume Moroz [AMN - defense planned in 2007 - F. Rouillier]

2. Overall Objectives

The main objective of the SALSA project is to solve systems of polynomial equations and inequations. We emphasize on algebraic methods which are more robust and frequently more efficient than purely numerical tools.

Polynomial systems have many applications in various scientific - academic as well as industrial - domains. However much work is yet needed in order to define specifications for the output of the algorithms which are well adapted to the problems.

The variety of these applications implies that our software needs to be robust. In fact, almost all problems we are dealing with are highly numerically unstable, and therefore, the correctness of the result needs to be guaranteed.

Thus, a key target is to provide software which are competitive in terms of efficiency but preserve certified outputs. Therefore, we restrict ourselves to algorithms which verify the assumptions made on the input, check the correctness of possible random choices done during a computation without sacrificing the efficiency. Theoretical complexity for our algorithms is only a preliminary step of our work which culminates with efficient implementations which are designed to solve significant applications.

A consequence of our way of working is that many of our contributions are related to applicative topics such as cryptography, error correcting codes, robotics and signal theory. We have to emphasize that these applied contributions rely on a long-term and global management of the project with clear and constant objectives leading to theoretical and deep advances.

3. Scientific Foundations

3.1. Introduction

For polynomial system solving, the mathematical specification of the result of a computation, in particular when the number of solutions is infinite, is itself a difficult problem [40], [41], [67], [66]. Sorting the most frequently asked questions appearing in the applications, one distinguishes several classes of problems which

are different either by their mathematical structure or by the significance that one can give to the word "solving".

Some of the following questions have a different meaning in the real case or in the complex case, others are posed only in the real case :

- zero-dimensional systems (with a finite number of complex solutions - which include the particular case of univariate polynomials); The questions in general are well defined (numerical approximation, number of solutions, etc) and the handled mathematical objects are relatively simple and well-known;
- parametric systems; They are generally zero-dimensional for almost all the parameters' values. The goal is characterize the solutions of the system (number of real solutions, existence of a parameterization, etc.) with respect to parameters' values.
- positive dimensional systems; For a direct application, the first question is the existence of zeros of a particular type (for example real, real positive, in a finite field). The resolution of such systems can be considered as a black box for the study of more general problems (semi-algebraic sets for example) and information to be extracted is generally the computation of a point per connected component in the real case.
- constructible and semi-algebraic sets; As opposed to what occurs numerically, the addition of constraints or inequalities complicates the problem. Even if semi-algebraic sets represent the basic object of the real geometry, their automatic "and effective study" remains a major challenge. To date, the state of the art is poor since only two classes of methods are existing :
 - the Cylindrical Algebraic Decomposition which basically computes a partition of the ambient space in cells where the signs of a given set of polynomials are constant;
 - deformations based methods that turn the problem into solving algebraic varieties.

The first solution is limited in terms of performances (maximum 3 or 4 variables) because of a recursive treatment variable by variable, the second also because of the use of a sophisticated arithmetic (formal infinitesimals).

- quantified formulas; deciding efficiently if a first order formula is valid or not is certainly one of the greatest challenges in "effective" real algebraic geometry. However this problem is relatively well encircled since it can always be rewritten as the conjunction of (supposed to be) simpler problems like the computation of a point per connected component of a semi-algebraic set.

As explained in some parts of this document, the unicity of the studied mathematical objects does not imply the unicity of the related algorithms.

The pressure is relatively strong on zero-dimensional systems since we enter directly in competition with numerical methods (Newton, homotopy, etc), semi-numerical methods (interval analysis, eigenvalues computations, etc.) and formal methods (geometrical resolution, resultant based strategies, XL algorithm in cryptography, etc.). The pressure is much less on the other subjects: except the groups working on the Cylindrical Algebraic Decomposition, very few studies with practical vocation are to be counted and even less software achievements.

The priorities we put on our algorithmic work are generally dictated by the applications. Thus, one finds the cutting stated above in the identified research topics of our project for the algorithmic part. For each of these goals, our work is to design the most efficient possible algorithms: there is thus a strong correlation between implementations and applications, but a significant part of work is dedicated to the identification of black-box allowing a modular approach of the problems. For example, the resolution of the zero-dimensional systems is a prerequisite for the algorithms treating of parametric or positive dimensional systems.

An essential class of black-box developed in the project does not appear directly in the absolute objectives counted above : the "algebraic or complex" resolutions. They are mostly reformulations, more algorithmically usable, of the studied systems. One distinguishes two categories of complementary objects :

- ideals representations; From a computational point of view these are the structures which are used in the first steps;
- varieties representations; The algebraic variety, or more generally the constructible or semi-algebraic set is the studied object.

To give a simple example, in \mathbb{C}^2 the variety $\{(0, 0)\}$ can be seen like the zeros set of more or less complicated ideals (for example, $\text{ideal}(X, Y)$, $\text{ideal}(X^2, Y)$, $\text{ideal}(X^2, X, Y, Y^3)$, etc). The entry which is given to us is a system of equations, i.e. an ideal. It is essential, in many cases, to understand the structure of this object to be able to correctly treat the degenerated cases. A striking example is certainly the study of the singularities. To take again the preceding example, the variety is not singular, but this cannot be detected by the blind application of the Jacobian criterion (one could wrongfully think that all the points are singular, contradicting, for example, Sard's lemma).

The basic tools that we develop and use to understand in an automatic way the algebraic and geometrical structures are on the one hand Gröbner bases (most known object used to represent an ideal without loss of information) and on the other hand triangular sets (effective way to represent the varieties).

On these two points, the pressure is strong since many teams work on these two objects. To date, our project however has a consequent advance in the computation of Gröbner bases (algorithms F_5 and F_7 of Faugère for Gröbner bases) and is a main contributor in the homogeneisation and comprehension of the triangular structures [41].

3.2. Gröbner basis and triangular sets

Participants: G. Ars, P. Aubry, M. Bardet, J.C. Faugère, S. Lacharte, D. Lazard, M. Safey El Din.

Let us denote by $K[X_1, \dots, X_n]$ the ring of polynomials with coefficients in a field K and indeterminates X_1, \dots, X_n and $S = \{P_1, \dots, P_s\}$ any subset of $K[X_1, \dots, X_n]$. A point $x \in \mathbb{C}^n$ is a zero of S if $P_i(x) = 0 \quad i \in [1 \dots s]$.

The ideal $\mathcal{J} = \langle P_1, \dots, P_s \rangle$ generated by P_1, \dots, P_s is the set of polynomials in $K[X_1, \dots, X_n]$ constituted by all the combinations $\sum_{k=1}^R P_k U_k$ with $U_k \in \mathbb{Q}[X_1, \dots, X_n]$. Since every element of \mathcal{J} vanishes at each zero of S , we denote by $V_C(S) = V_C(I) = \{x \in C^n \mid p(x) = 0 \forall p \in \mathcal{J}\}$ (resp. $V_R(S) = V_R(I) = V_C(I) \cap \mathbb{R}^n$), the set of complex (resp. real) zeros of S , where R is a real closed field containing K and C its algebraic closure.

One Gröbner basis' main property is to provide an algorithmic method for deciding if a polynomial belongs or not to an ideal through a reduction function denoted "Reduce" from now.

If G is a Gröbner basis of an ideal $\mathcal{J} \subset \mathbb{Q}[X_1, \dots, X_n]$ for any monomial ordering $<$.

- a polynomial $p \in \mathbb{Q}[X_1, \dots, X_n]$ belongs to \mathcal{J} if and only if $\text{Reduce}(p, G, <) = 0$,
- $\text{Reduce}(p, G, <)$ does not depend on the order of the polynomials in the list G , thus, this is a canonical reduced expression modulus \mathcal{J} , and the Reduce function can be used as a *simplification* function.

Gröbner bases are computable objects. The most popular method for computing them is Buchberger's algorithm ([50], [49]). It has several variants and it is implemented in most of general computer algebra systems like Maple or Mathematica. The computation of Gröbner bases using Buchberger's original strategies has to face to two kind of problems :

- (A) arbitrary choices : the order in which are done the computations has a dramatic influence on the computation time;

- (B) useless computations : the original algorithm spends most of its time in computing 0.

For problem (A), J.C. Faugère proposed ([55] - algorithm F_4) a new generation of powerful algorithms ([55]) based on the intensive use of linear algebra technics. In short, the arbitrary choices are left to computational strategies related to classical linear algebra problems (matrix inversions, linear systems, etc.).

For problem (B), J.C. Faugère proposed ([54]) a new criterion for detecting useless computations. Under some regularity conditions on the system, it is now proved that the algorithm do never perform useless computations.

A new algorithm named F_5 was built using these two key results. Even if it still computes a Gröbner basis, the gap with existing other strategies is consequent. In particular, due to the range of examples that become computable, Gröbner basis can be considered as a reasonable computable object in large applications.

We pay a particular attention to Gröbner bases computed for elimination orderings since they provide a way of "simplifying" the system (a equivalent system with a structured shape). For example, a lexicographic Gröbner basis has always the following shape :

$$\left\{ \begin{array}{l} f(X_1) = 0 \\ f_2(X_1, X_2) = 0 \\ \vdots \\ f_{k_2}(X_1, X_2) = 0 \\ f_{k_2+1}(X_1, X_2, X_3) = 0 \\ \vdots \\ f_{k_{n-1}+1}(X_1, \dots, X_n) = 0 \\ \vdots \\ f_{k_n}(X_1, \dots, X_n) = 0 \end{array} \right. .$$

(some of the polynomials may be identically null). A well known property is that the zeros of the first non null polynomial define the Zariski closure (classical closure in the case of complex coefficients) of the projection on the coordinate's space associated with the smallest variables.

A triangular set is a system with the following shape :

$$\left\{ \begin{array}{l} t_1(X_1) = 0 \\ t_2(X_1, X_2) = 0 \\ \vdots \\ t_n(X_1, \dots, X_n) = 0 \end{array} \right.$$

(some polynomials may be identically null).

Such kinds of systems are algorithmically easy to use, for computing numerical approximations of the solutions in the zero-dimensional case or for the study of the singularities of the associated variety (triangular minors in the Jacobian matrices). Except if they are linear, algebraic systems cannot, in general, be rewritten as a single triangular set, one speaks then of decomposition of the systems in several triangular sets.

Triangular sets appear under various names in the field of algebraic systems. In 1932 J.F. Ritt ([76]) introduced them as characteristic sets for prime ideals in the context of differential algebra. His constructive algebraic tools were adapted by W.T. Wu in the late seventies for geometric applications. Wu presented an algorithm for computing characteristic sets of finite polynomial sets which do not generate necessarily prime ideals ([98], [99]). With Wu, several authors such S.C. Chou, X.S. Gao, G. Gallo, B. Mishra, D. Wang then developed this approach to make it more efficient.

In 1991, Lazard [68] and Kalkbrener [65] presented triangular decomposition algorithms with nicer properties for their outputs, based on additional requirements for the triangular sets and a generalisation of the gcd of univariate polynomials over a product of fields.

The concept of regular chain introduced in [65] and [100] is adapted for recursive computations in a univariate way and provides a membership test and a zero-divisor test for the strongly unmixed dimensional ideal it defines. Kalkbrenner defined regular triangular sets and showed how to decompose algebraic varieties as a union of Zariski closures of zeros of regular triangular sets. Gallo showed that the principal component of a triangular decomposition can be computed in $O(d^{O(n^2)})$ (n = number of variables, d =degree in the variables). During the 90s, implementations of various strategies of decompositions multiply, but they drain relatively heterogeneous specifications.

Following Kalkbrenner's work, Aubry presented an algorithm for decomposing the radical of an ideal into separable regular chains that define radical strongly unmixed dimensional ideals ([40]).

P. Aubry and D. Lazard contributed to the homogenisation of the work completed in this field by proposing a series of specifications and definitions gathering the whole of former work [41]. Two essential concepts for the use of these sets (regularity, separability) at the same time allow from now on to establish a simple link with the studied varieties and to specify the computed objects precisely.

For $p \in K[X_1, \dots, X_n] \setminus \mathbb{Q}$, we denote by $\text{mvar}(p)$ (and we call *main variable* of p) the greatest variable appearing in p w.r.t. a fixed lexicographic ordering.

- h_i the leading coefficient of t_i (when $t_i \neq 0$ it is seen as a univariate polynomial in its main variable), and $h = \prod_{i=1, t_i \neq 0}^n h_i$.
- $s_i = \frac{\partial t_i}{\partial X_i}$ the separant of t_i (when $t_i \neq 0$) $s = \prod_{i=1, t_i \neq 0}^n s_i$.
- $\text{sat}(T) = \langle T \rangle : h^\infty = \{p \in K[X_1, \dots, X_n] \mid \exists m \in \mathbb{N}, h^m p \in \langle T \rangle\}$; the variety of T is $\mathbf{V}(\text{sat}(T))$ and we have $\mathbf{V}(T) \setminus \mathbf{V}(h) = \mathbf{V}(\text{sat}(T))$ (elementary property of localization).

A triangular set $T = (t_1, \dots, t_n) \subset K[X_1, \dots, X_n]$ is said to be regular (resp. separable) if $\forall i \in \{1, \dots, n\}$ such that $t_i \neq 0$, the normalization of its initial h_i (resp. of its separant s_i) is a non zero polynomial.

One can always decompose a variety as the union of the varieties of regular and separable triangular sets ([40], [41]):

$$V_C = \bigcup_i \mathbf{V}(\text{sat}(T_i)). \quad (1)$$

A remarkable and fundamental property in the use we have of the triangular sets is that the ideals $\text{sat}(T_i)$, for regular and separable triangular sets, are radical and equi-dimensional. These properties are essential for some of our algorithms. For example, having radical and equi-dimensional ideals allows us to compute straightforwardly the singular locus of a variety by canceling minors of good dimension in the Jacobian matrix of the system. This is naturally a basic tool for some algorithms in real algebraic geometry [42], [85], [16].

Triangular sets based technics are efficient for specific problems like computing Galois ideals [43], but the implementations of direct decompositions into triangular sets do not currently reach the level of efficiency of Gröbner bases in terms of computable classes of examples. Anyway, our team benefits from the progress carried out in this last field since we currently perform decompositions into regular and separable triangular sets through lexicographical Gröbner bases computations (the process provides in the meantime Gröbner bases of the ideals $\text{sat}(T_i)$).

3.3. Zero-dimensional systems

Participants: P. Aubry, J.C. Faugère, A. Hashemi, D. Lazard, F. Rouillier, P. Trébuchet.

A system is zero-dimensional if the set of the solutions in an algebraically closed field is finite. In this case, the set of solutions does not depend on the chosen algebraically closed field.

Such a situation can easily be detected on a Gröbner basis for any admissible monomial ordering.

These systems are mathematically particular since one can systematically bring back to solve linear algebra problems. More precisely, the algebra $K[X_1, \dots, X_n]/I$ is in fact a K - vector space of dimension equal to the number of complex roots of the system (counted with multiplicities). We chose to exploit this structure. Accordingly, computing a base of $K[X_1, \dots, X_n]/I$ is essential. A Gröbner basis gives a canonical projection from $K[X_1, \dots, X_n]$ to $K[X_1, \dots, X_n]/I$, and thus provides a base of the quotient algebra and many other informations more or less straightforwardly (number of complex roots for example).

The use of this vector-space structure is well known and at the origin of the one of the most known algorithms of the field ([56]) : it allows to deduce, starting from a Gröbner basis for any ordering, a Gröbner base for any other ordering (in practice, a lexicographic basis, which are very difficult to calculate directly). It is also common to certain semi-numerical methods since it allows to obtain quite simply (by a computation of eigenvalues for example) the numerical approximation of the solutions (this type of algorithms is developed, for example, in the INRIA Galaad project).

Contrary to what is written in a certain literature, the computation of Gröbner bases is not "doubly exponential" for all the classes of problems. In the case of the zero-dimensional systems, it is even shown that it is simply exponential in the number of variables, for a degree ordering and for the systems without zeros at infinity. Thus, an effective strategy consists in computing a Gröbner basis for a favourable ordering and then to deduce, by linear algebra technics, a Gröbner base for a lexicographic ordering [56].

The case of the zero-dimensional systems is also specific for triangular sets. Indeed, in this particular case, we have designed algorithms that allow to compute them efficiently [69] starting from a lexicographic Gröbner basis. Note that, in the case of zero-dimensional systems, regular triangular sets are Gröbner bases for a lexicographical order.

Many teams work on Gröbner bases and some use triangular sets in the case of the zero-dimensional systems, but up to our knowledge, very few continue the work until a numerical resolution and even less tackle the specific problem of computing the real roots. It is illusory, in practice, to hope to obtain numerically and in a reliable way a numerical approximation of the solutions straightforwardly from a lexicographical basis and even from a triangular set. This is mainly due to the size of the coefficients in the result (rational number).

Our specificity is to carry out the computations until their term thanks to two types of results :

- the computation of the Rational Univariate Representation [81] : we shown that any zero-dimensional system, depending on variables X_1, \dots, X_n , can systematically be rewritten, without loss of information (multiplicities, real roots), in the form $f(T) = 0, X_i = g_i(T)/g(T), i = 1 \dots n$ where the polynomials f, g, g_1, \dots, g_n have coefficients in the same ground field as those of the system and where T is a new variable (independant from X_1, \dots, X_n).
- efficient algorithms for solving (real roots isolation and counting) univariate polynomials [84], [73].

Thus, the use of innovative algorithms for Gröbner bases computations [55], [54], Rational Univariate representations ([56] for the "shape position" case and [81] for the general case), allows to use zero-dimensional solving as sub-task in other algorithms.

3.4. Positive-dimensional and parametric systems

Participants: P. Aubry, J.C. Faugère, A. Hashemi, D. Lazard, G. Moroz, F. Rouillier, M. Safey El Din.

When a system is **positive dimensional** (with an infinite number of complex roots), it is no more possible to enumerate the solutions. Therefore, the solving process reduces to decomposing the set of the solutions into subsets which have a well-defined geometry. One may perform such a decomposition from an algebraic point of view or from a geometrical one, the latter meaning not taking the multiplicities into account (structure of primary components of the ideal is lost).

Although there exist algorithms for both approaches, the algebraic point of view is presently out of the possibilities of practical computations, and we restrict ourselves to geometrical decompositions.

When one studies the solutions in an algebraically closed field, the decompositions which are useful are the equi-dimensional decomposition (which consists in considering separately the isolated solutions, the curves, the surfaces, ...) and the prime decomposition (decomposes the variety into irreducible components). In practice, our team works on algorithms for decomposing the system into *regular separable triangular sets*, which corresponds to a decomposition into equi-dimensional but not necessarily irreducible components. These irreducible components may be obtained eventually by using polynomial factorization.

However, in many situations one is looking only for real solutions satisfying some inequalities ($P_i > 0$ or $P_i \geq 0$)¹. In this case, there are various kinds of decompositions besides the above ones: connected components, cellular or simplicial decompositions, ...

There are general algorithms for such tasks, which rely on Tarski's quantifier elimination. Unfortunately, these problems have a very high complexity, usually doubly exponential in the number of variables or the number of blocks of quantifiers, and these general algorithms are intractable. It follows that the output of a solver should be restricted to a partial description of the topology or of the geometry of the set of solutions, and our research consists in looking for more specific problems, which are interesting for the applications, and which may be solved with a reasonable complexity.

We focus on 2 main problems :

- computing one point on each connected components of a semi-algebraic set;
- solving systems of equalities and inequalities depending on parameters.

3.4.1. Critical point methods

The most widespread algorithm computing sampling points in a semi-algebraic set is the Cylindrical Algebraic Decomposition Algorithm due to Collins [51]. With slight modifications, this algorithm also solves the problem of Quantifier Elimination. It is based on the recursive elimination of variables one after another ensuring nice properties between the components of the studied semi-algebraic set and the components of semi-algebraic sets defined by polynomial families obtained by the elimination of variables. It is doubly exponential in the number of variables and its best implementations are limited to problems in 3 or 4 variables.

Since the end of the eighties, alternative strategies (see [61][62][63][46][47]) with a single exponential complexity in the number of variables have been developed. They are based on the progressive construction of the following subroutines:

- (a) solving zero-dimensional systems: this can be performed by computing a Rational Univariate Representation (see [81]);
- (b) computing sampling points in a real hypersurface: after some infinitesimal deformations, this is reduced to problem (a) by computing the critical locus of a polynomial mapping reaching its extrema on each connected component of the real hypersurface;

¹In the zero-dimensional case, inequations and inequalities are usually taken into account only at the end of the computation, to eliminate irrelevant solutions.

- (c) computing sampling points in a real algebraic variety defined by a polynomial system: this is reduced to problem (b) by considering the sum of squares of the polynomials;
- (d) computing sampling points in a semi-algebraic set: this is reduced to problem (c) by applying an infinitesimal deformation.

On the one hand, the relevance of this approach is based on the fact that its complexity is asymptotically optimal. On the other hand, some important algorithmic developments have been necessary to obtain efficient implementations of subroutines (b) and (c).

During the last years, we focused on providing efficient algorithms solving the problems (b) and (c). The used method rely on finding a polynomial mapping reaching its extrema on each connected component of the studied variety such that its critical locus is zero-dimensional. For example, in the case of a smooth hypersurface whose real counterpart is compact choosing a projection on a line is sufficient. This method is called in the sequel the critical point method. We started by studying problem (b) [82].

Even if we showed that our solution may solve new classes of problems ([83]), we have chosen to skip the reduction to problem (b), which is now considered as a particular case of problem (c), in order to avoid an artificial growth of degree and the introduction of singularities and infinitesimals.

Putting the critical point method into practice in the general case requires to drop some hypotheses. First, the compactness assumption, which is in fact intimately related to an implicit properness assumption, has to be dropped. Second, algebraic characterizations of critical loci are based on assumptions of non-degeneracy on the rank of the Jacobian matrix associated to the studied polynomial system. These hypotheses are not satisfied as soon as this system defines a non-radical ideal and/or a non equi-dimensional variety, and/or a non-smooth variety. Our contributions consist in overcoming efficiently these obstacles ([87], [77]) and several strategies have been developed [42], [85],[16].

The properness assumption can be dropped by considering the square of a distance function to a generic point instead of a projection function: indeed each connected component contains at least a point minimizing locally this function. Performing a radical and equi-dimensional decomposition of the ideal generated by the studied polynomial system allows to avoid some degeneracies of its associated Jacobian matrix. At last, the recursive study of overlapped singular loci allows to deal with the case of non-smooth varieties. These algorithmic issues allow to obtain a first algorithm [42] with reasonable practical performances.

Since projection functions are linear while the distance function is quadratic, computing their critical points is easier. Thus, we have also investigated their use. A first approach [85] consists in studying recursively the critical locus of projection functions on overlapped affine subspaces containing coordinate axes combined with the study of their set of non-properness. A more efficient one [16], avoiding the study of sets of non-properness is obtained by considering iteratively projections on *generic* affine subspaces restricted to the studied variety and fibers on arbitrary points of these subspaces intersected with the critical locus of the corresponding projection. The underlying algorithm is the most efficient we obtained.

The algorithms of [42], [85] are provided in the Maple Library RAGLib. It is built upon the softwares Gb and RS. It contains functionalities for computing sample points in real algebraic varieties, semi-algebraic sets defined by non-strict inequalities, and the radical and equi-dimensional decomposition of an ideal. The experimental version of the algorithm provided in [86] will be included in the next release which is in preparation.

In terms of complexity, we have proved in [39] that when the studied polynomial system generates a radical ideal and defines a smooth algebraic variety, the output of our algorithms is smaller than what could be expected by applying the classical Bézout bound and than the output of the previous algorithms. This has also given new upper bounds on the number of connected components of a smooth real algebraic variety which improve the classical Thom-Milnor bound. The technique we used, also allows to prove that the degree of the critical locus of a projection function is inferior or equal to the degree of the critical locus of a distance function. Finally, it shows how to drop the assumption of equidimensionality required in the aforementioned

algorithms. This technique is based on the study of the Lagrange's system associated to a polynomial family, the study of its bi-homogeneous structure, and a strong bi-homogeneous Bézout theorem we proved which allows to bound the sum of the degrees of all the isolated primary components of an ideal generated by a bi-homogeneous system.

3.4.2. Parametric systems

Most of the applications we recently solved (celestial mechanics, cuspidal robots, statistics, etc.) require the study of semi-algebraic systems depending on parameters. Although we covered these subjects in an independent way, some general algorithms for the resolution of this type of systems can be proposed from these experiments.

The general philosophy consists in studying the generic solutions independently from algebraic subvarieties (which we call from now on discriminant varieties) of dimension lower than the semi-algebraic set considered. The study of the varieties thus excluded can be done separately to obtain a complete answer to the problem, or is simply neglected if one is interested only in the generic solutions, which is the case in some applications.

We recently proposed a new framework for studying basic constructible (resp. semi-algebraic) sets defined as systems of equations and inequations (resp. inequalities) depending on parameters. Let's consider the basic semi-algebraic set

$$\mathcal{S} = \{x \in \mathbb{R}^n, p_1(x) = 0, \dots, p_s(x) = 0, f_1(x) > 0, \dots, f_s(x) > 0\}$$

and the basic constructible set

$$\mathcal{C} = \{x \in \mathbb{C}^n, p_1(x) = 0, \dots, p_s(x) = 0, f_1(x) \neq 0, \dots, f_s(x) \neq 0\}$$

where p_i, f_j are polynomials with rational coefficients.

- $[U, X] = [U_1, \dots, U_d, X_{d+1}, \dots, X_n]$ is the set of *indeterminates* or variables, while $U = [U_1, \dots, U_d]$ is the set of *parameters* and $X = [X_{d+1}, \dots, X_n]$ the set of *unknowns*;
- $\mathcal{E} = \{p_1, \dots, p_s\}$ is the set of polynomials defining the equations;
- $\mathcal{F} = \{f_1, \dots, f_t\}$ is the set of polynomials defining the inequations in the complex case (resp. the inequalities in the real case);
- For any $u \in \mathbb{C}^d$ let φ_u be the specialization $U \rightarrow u$;
- $\Pi_U : \mathbb{C}^n \rightarrow \mathbb{C}^d$ denotes the canonical projection on the parameter's space $(u_1, \dots, u_d, x_{d+1}, \dots, x_n) \rightarrow (u_1, \dots, u_d)$;
- Given any ideal I we denote by $\mathbf{V}(I) \subset \mathbb{C}^n$ the associated (algebraic) variety. If a variety is defined as the zero set of polynomials with coefficients in \mathbb{Q} we call it a \mathbb{Q} -algebraic variety; we extend naturally this notation in order to talk about \mathbb{Q} -irreducible components, \mathbb{Q} -Zariski closure, etc.
- for any set $\mathcal{V} \subset \mathbb{C}^n$, $\overline{\mathcal{V}}$ will denote its \mathbb{C} -Zariski closure in \mathbb{C}^n .

In most applications, $\mathbf{V}(\langle \varphi_u(\mathcal{E}) \rangle)$ as well as $\varphi_u(\mathcal{C}) = \Pi_U^{-1}(u) \cap \mathcal{C}$ are finite and not empty for almost all parameter's u . Most algorithms that study \mathcal{C} or \mathcal{S} (number of real roots w.r.t. the parameters, parameterizations of the solutions, etc.) compute in any case a \mathbb{Q} -Zariski closed set $W \subset \mathbb{C}^d$ such that for any $u \in \mathbb{C}^d \setminus W$, there exists a neighborhood \mathcal{U} of u with the following properties :

- $(\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}, \Pi_U)$ is an analytic covering of \mathcal{U} ; this implies that the elements of \mathcal{F} do not vanish (and so have constant sign in the real case) on the connected components of $\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}$;

We recently show that the parameters' set such that there doesn't exist any neighborhood \mathcal{U} with the above analytic covering property is a \mathbb{Q} -Zariski closed set which can exactly be computed. We name it the *minimal discriminant variety of \mathcal{C} with respect to Π_U* and propose also a definition in the case of non generically zero-dimensional systems [32], [29].

Being able to compute the minimal discriminant variety allows to simplify the problem depending on n variables to a similar problem depending on d variables (the parameters) : it is sufficient to describe its complementary in the parameters' space (or in the closure of the projection of the variety in the general case) to get the full information about the generic solutions (here generic means for parameters' values outside the discriminant variety).

Then being able to describe the connected components of the complementary of the discriminant variety in \mathbb{R}^d becomes a main challenge which is strongly linked to the work done on positive dimensional systems. Moreover, rewriting the systems involved and solving zero-dimensional systems are major components of the algorithms we plan to build up.

We currently propose several computational strategies. An a priori decomposition into equidimensional components as zeros of radical ideals simplifies the computation and the use of the discriminant varieties. This preliminary computation is however sometimes expensive, so we are developing adaptive solutions where such decompositions are called by need. The main progress is that the resulting methods are fast on easy problems (generic) and slower on the problems with strong geometrical contents.

We also defined (large) *discriminant varieties of \mathcal{C} with respect to Π_U* as being any \mathbb{Q} -Zariski closed set W containing the minimal discriminant variety of \mathcal{C} with respect to Π_U (the minimal discriminant variety is the smallest discriminant variety and it is uniquely defined).

The existing implementations of algorithms able to "solve" (to get some information about the roots) parametric systems do all compute (directly or indirectly) discriminant varieties but none computes optimal objects (strict discriminant variety). The consequence is that the output (case distinctions w.r.t. parameters' values) are huge compared with the results we can provide.

The Cylindrical Algebraic Decomposition adapted to $\mathcal{E} \cup \mathcal{F}$ [51] computes explicitly a discriminant variety of \mathcal{C} as soon as the recursive projection steps w.r.t. X_{d+1}, \dots, X_n are done first. A discriminant variety will then be defined by the union of the varieties associated with the polynomials obtained after the $n - d$ -th projection step. It is far from being optimal with respect to the number of elements in the induced partition; in fact it contains at least, a discriminant variety for any system of equalities and inequalities that can be constructed with the polynomials of $\mathcal{E} \cup \mathcal{F}$. The same remark applies for recursive methods based on univariate resultant.

Algorithms based on "Comprehensive Gröbner bases" [93], [94], [92], compute also (implicitly or explicitly) discriminant varieties. In the case of parametric systems, such a discriminant variety contains the parameters' values for which a Gröbner basis do not specialize properly. Again, it is far from being optimal since it contains the parameter's values where the staircase varies, which depend on the strategy used (for example the choice of a monomial ordering).

Methods that compute parameterizations of the solutions (see [89] for example) compute also discriminant varieties. Again, these are not optimal since they depend on the strategy used. Precisely, the result depends on an arbitrary chosen "separating element" $t \in \mathbb{Q}[X_{d+1}, \dots, X_n]$ which is a polynomial (often a linear form) that induces, for almost all the parameters, an injective map from $\mathbf{V}(\langle \mathcal{E} \rangle)$ to C^d . If p is the minimal polynomial of t viewed as an element of $\mathbb{Q}(U_1, \dots, U_d)[X_{d+1}, \dots, X_n]/\sqrt{\langle \mathcal{E} \rangle}$, the induced discriminant variety contains all the parameters' values such that the leading coefficient of p or its discriminant vanishes; this includes, in particular, the parameter's values such that t does not separate the zeros of the corresponding specialization of $\sqrt{\langle \mathcal{E} \rangle}$.

4. Application Domains

4.1. Panorama

Applications are fundamental for our research for several reasons.

The first one is that they are the only source of fair tests for the algorithms. In fact, the complexity of the solving process depends very irregularly of the problem itself. Therefore, random tests do not give a right idea of the practical behavior of a program, and the complexity analysis, when possible, does not necessarily provide realistic information.

A second reason is that, as quoted above, we need real world problems to determine which specifications of algorithms are really useful. Conversely, it is frequently by solving specific problems through ad hoc methods that we found new algorithms with general impact.

Finally, obtaining successes with problems which are intractable by the other known approaches is the best proof for the quality of our work.

On the other hand, there is a specific difficulty. The problems which may be solved with our methods may be formulated in many different ways, and their usual formulation is rarely well suited for polynomial system solving or for exact computations. Frequently, it is not even clear that the problem is purely algebraic, because researchers and engineers are used to formulate them in a differential way or to linearize them.

Therefore, our software may not be used as black boxes, and we have to understand the origin of the problem in order to translate it in a form which is well suited for our solvers.

It follows that many of our results, published or in preparation, are classified in scientific domains which are different from ours, like cryptography, error correcting codes, robotics, signal processing, statistics or biophysics.

4.2. Cryptography

The idea of using multivariate (quadratic) equations as a basis for building public key cryptosystems appeared with the Matsumoto-Imai cryptosystem. This system was first broken by Patarin and, shortly after, Patarin proposed to repair it and thus devised the hidden field equation (HFE) cryptosystem.

The basic idea of HFE is simple: build the secret key as a univariate polynomial $S(x)$ over some (big) finite field (often $\text{GF}(2^n)$). Clearly, such a polynomial can be easily evaluated; moreover, under reasonable hypotheses, it can also be “inverted” quite efficiently. By inverting, we mean finding any solution to the equation $S(x) = y$, when such a solution exists. The secret transformations (decryption and/or signature) are based on this efficient inversion. Of course, in order to build a cryptosystem, the polynomial S must be presented as a public transformation which hides the original structure and prevents inversion. This is done by viewing the finite field $\text{GF}(2^n)$ as a vector space over $\text{GF}(2)$ and by choosing two linear transformations of this vector space L_1 and L_2 . Then the public transformation is the composition of L_1 , S and L_2 . Moreover, if all the terms in the polynomial $S(x)$ have Hamming weight 2, then it is obvious that all the (multivariate) polynomials of the public key are of degree two.

By using fast algorithms for computing Gröbner bases, it was possible to break the first HFE challenge [64] (real cryptographic size 80 bits and a symbolic prize of 500 US\$) in only two days of CPU time. More precisely we have used the $F_5/2$ version of the fast F_5 algorithm for computing Gröbner bases (implemented in C). The algorithms available up to now (Buchberger) were extremely slow and could not have been used to break the code (they should have needed at least a few centuries of computation). The new algorithm is thousands of times faster than previous algorithms. Several matrices have to be reduced (Echelon Form) during the computation: the biggest one has no less than 1.6 million columns, and requires 8 gigabytes of memory. Implementing the algorithm thus required significant programming work and especially efficient memory management.

A new result is that the weakness of the systems of equations coming from HFE instances can be *explained* by the algebraic properties of the secret key (work presented at Crypto 2003 in collaboration with A. Joux).

From this study we are able to predict the maximal degree occurring in the Gröbner basis computation, so that we can establish precisely the complexity of the Gröbner attack and compare it with the theoretical bounds.

Since it is easy to transform many cryptographic problems into polynomial equations, our group is in a position to apply this general method to other cryptosystems. Thus we have a new general cryptanalysis approach, called algebraic cryptanalysis. The team is currently testing the robustness of cryptosystems based on nonlinear filter generators (with G. Ars, J.-C. Faugère) and in collaboration with the Codes team and DGA (Celar).

Another relevant tool in the study of cryptographic problems is the LLL algorithm which is able to compute in polynomial time a “good” approximation for the shortest vector problem. Since a Gröbner basis can be seen as the set of smallest polynomials in an ideal with respect to the divisibility of leading terms, it is natural to compare both algorithms: an interesting link between LLL (polynomial version) and Gröbner bases was suggested by a member of our group.

A standard algorithm for implementing the arithmetic of Jacobian groups of curves is LLL. By replacing LLL by the FGLM algorithm we establish a new structure theorem for Gröbner bases; consequently, on a generic input we were able to establish explicit and optimized formulas for basic arithmetic operations in the Jacobian groups of C_{34} curves [20].

As an application of the LLL algorithm we have presented [44], an algorithm for converting a Gröbner basis of an ideal with respect to any given ordering into a Gröbner basis with respect to any other ordering. This algorithm is based on a modified version of the LLL algorithm. In the worst case, the theoretical complexity of this algorithm is not necessarily better than the complexity of the FGLM algorithm; but when the output (the final Gröbner basis) is small this algorithm is experimentally more efficient.

4.3. Parallel robots

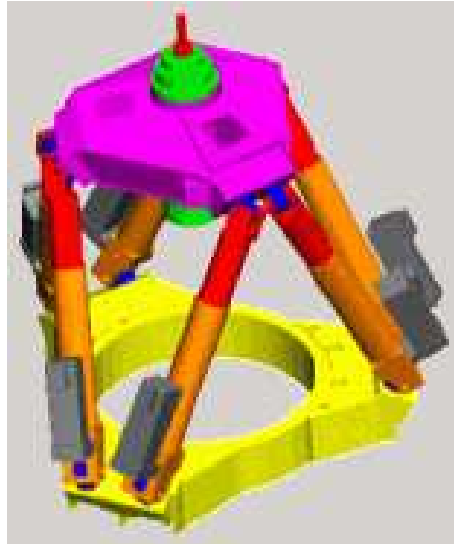


Figure 1. An Hexapod

The (parallel) manipulators we study are general parallel robots: the hexapods are complex mechanisms made up of six (often identical) kinematic chains, of a base (fixed rigid body including six joints or articulations) and of a platform (mobile rigid body containing six other joints).

The design and the study of parallel robots require the resolution of direct geometrical models (computation of the absolute coordinates of the joints of the platform knowing the position and the geometry of the base, the geometry of the platform as well as the distances between the joints of the kinematic chains at the base and the platform) and inverse geometrical models (distances between the joints of the kinematic chains at the base and the platform knowing the absolute positions of the base and the platform).

Since the inverse geometrical models can be easily solved, we focus on the resolution of the direct geometrical models.

The study of the direct geometrical model is a recurrent activity for several members of the project. One can say that the progress carried out in this field illustrates perfectly the evolution of the methods for the resolution of algebraic systems. The interest carried on this subject is old. The first work in which the members of the project took part in primarily concerned the study of the number of (complex) solutions of the problem [71], [70]. The results were often illustrated by Gröbner bases done with Gb software. One of the remarkable points of this study is certainly the classification suggested in [60]. The next efforts were related to the real roots and the effective computation of the solutions [79]. The studies then continued following the various algorithmic progresses, until the developed tools made possible to solve non-academic problems. In 1999, the various efforts were concretized by an industrial contract with the SME CMW (*Constructions Mécaniques des Vosges-Marioni*) for studying a robot dedicated to machine tools.



Figure 2. The CMW300

We conceived, in collaboration with the COPRIN project, a prototype of simulator for validating a fixed trajectory, i.e. :

- check that the trajectory is nonsingular for a series of functions modelizing the length of the legs : let us recall that for given values of the legs' length, there exists up to 40 possible positions. To check that the trajectory is nonsingular, one must ensure, for example, that 2 possible trajectories do not intersect (in which case the robot cannot be controlled);
- measure without ambiguity the difference between two trajectories corresponding to different legs' length functions (this is a tool for checking "numerical" singularities).

This tool is single in the world : concurrent solutions exist but can treat only particular robots (plan, symmetrical, less joints, etc). It is necessary to know to how to solve the general case because a small modification of the design parameters' (unavoidable in practice) has serious consequences on the behavior of the robot. For example, the theoretical robot we use (based on the left-hand parallel manipulator due to

J.-P. Merlet) for our study admits at most 36 solutions for the direct geometrical model (either up to 36 possible trajectories for fixed length legs' functions), whereas the presently built robot (with small errors on the positions of the joints) has up to 40 possible positions.

The main algorithmic tool present in this simulator (partially presented in [78]) is a hybrid method (mixing computer algebra, numerical computation and interval analysis) for the resolution of the direct geometrical model.

A part of the work was to develop a semi-numerical method (based on Newton's method), powerful in terms of computation time (4000 computations per minute) and certified, i.e. always returning a correct result: the answer is either a set of numerical values with a certified precision or a failure message. The strategy used combines interval arithmetics and convergence results. The failure remains exceptional (less than 10 percent of the practical problems) and, when it occurs, the result is obtained using a special version of the F_4 algorithm for Gröbner bases computation and an optimized version (adapted to that particular case of systems) of the Rational Univariate Representation algorithm.

Our simulator has been used to diagnose the problems related to the solutions currently employed (CAD, look-ahead, algorithms for interpolating the trajectories, etc).

4.4. Serial robots

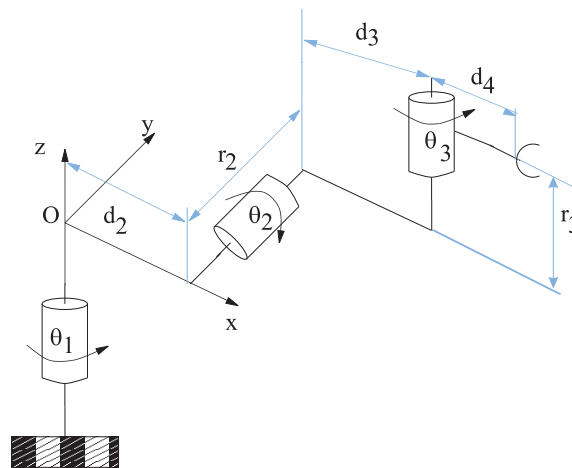


Figure 3. A serial robot with 3 D.O.F.

Industrial robotic (serial) manipulators with 3 degrees of freedom are currently designed with very simple geometric rules on the designed parameters, the ratios between them are always of the same kind. In order to enlarge the possibilities of such manipulators, it may be interesting to relax some constraints on the parameters.

However, the diversity of the tasks to be done carries out to the study of other types of robots whose parameters of design differ from what is usual and which may have new properties, like stability or existence of new kinds of trajectories.

An important difficulty slows down the industrial use of such new robots : recent studies ([95], [97], [96] and [75]) showed that they may have a behavior which is qualitatively different from those of the robots currently used in industry and allows new changes of posture. These robots, called cuspidal, cannot be controlled like the others. The majority of the robots are in fact cuspidal: the industrial robots currently on the market form a very restricted subclass of all the possible robots.

A full characterization of all the cuspidal robots is of a great interest for the designer and the user. Such a project forms part of a current tendency in robotics which consists in designing a robot in order that its performances are optimal for a given application while preserving the possibility of using it for another task, that is to say to specialize it to the maximum for an application in order to reduce its cost and to increase its operational safety.

The study of the behavior at a change of posture is identical, from the computer algebra point of view, to solving a system of equalities and inequalities depending on three or four parameters which correspond to the design parameters of this kind of robots. The method we basically used was "ad hoc", and no known automatic computer algebra methods were able to solve completely the problem before the work done in collaboration with COPRIN (INRIA Sophia), IRMAR (University of Rennes I) and IRRCyN (CNRS - Nantes) teams.

From a robotic point of view, the result obtained is a full classification of a class of serial robots with three degrees of freedom according to their cuspidal character. Since then, toward the end of the Maths-Stic project "Robots Cuspidaux", these results were simplified and analyzed to allow a better description of the workspace of such mechanisms.

The computations done for this application were critical also for the development of the general and systematic methods for solving parametric systems. We have shown that these general methods can now be used in place of ad hoc computations to calculate the same classification and a recent experiment even shows that they allow to relax one more parameter and thus to solve a more general problem.

4.5. Signal Processing

Some problems in signal theory are naturally formulated in terms of algebraic systems. In [59], we had studied the Kovacevic-Vetterli's family of filters banks. To be used for image compression, a wavelet transformation must be defined by a function having a maximum of partial derivative that vanishes at the corners of the image. These conditions can be translated to polynomial systems that can be solved with our methods. We showed that to get physically acceptable solutions, it was necessary to choose the number of conditions so that the solutions' space is of dimension 0, 2 or 4 (according to the size of the filter). This result (parametric family of filters) is subject to a patent [53]. To exploit these filters in practice, it remains to choose the best transformation, according to non-algebraic criteria, which is easily done with traditional tools for optimization (with a reduced number of variables).

As for most of applications on which we work, it took more than three years to obtain concrete results bringing real practical progress (the results mentioned in [80] are partial), and still a few years more to be able to disseminate information towards our community [58]. Our software tools are now used to solve nearby problems [72].

Our activity in signal processing started again a few months ago through a collaboration with the APICS project (collaboration with F. Seyfert) on the synthesis and identification of hyperfrequency filters made of coupled resonant cavities. It is now part of our research goals.

5. Software

5.1. GC

Participants: J.C. Faugère [contact], F. Rouillier.

One specificity in computer algebra is to manipulate huge objects with a size that varies along the algorithm. Having a specific memory manager, adapted to the objects handled in the various implementations is thus essential. Based on one concept suggested by J-C. Faugère in his PhD thesis, several versions implemented in C are used in different software packages of the project (GB, FGB, RS) as well as in implementations due to collaborators (F. Boulier - LIFL). The various suggested implementations are very simple and it seems preferable to precisely describe the process and its use in some key situations than to propose a standardized implementation as a library.

5.2. UDX

Participants: F. Rouillier [contact], J.C. Faugère.

See <http://fgbrs.lip6.fr/salsa/>. The result of a Gröbner basis computation may be huge and is the main input of our high-level algorithms. The time needed for transferring such an object using ascii files or pipes may be greater than the computation time.

UDX is a software for binary data exchange. It was initially developed to show the power of a new protocol, object of a patent by INRIA and UPMC [57]. The resulting code, written in ANSI C (9500 lines), is very portable and very efficient, even when the patented protocol is not used. UDX is composed of five independent modules:

- base: optimized system of buffers and synchronization of the input and output channels;
- supports: read/write operations on various supports (sockets, files, shared memory, etc.);
- protocols: various exchange protocols (patented protocol, XDR, etc.);
- exchange of composite types: floating-point numbers (simple and double precision), multiprecision integers, rational numbers;
- interfaces: user interfaces implementing high level callings to the four other modules.

5.3. MPFI

Participants: F. Rouillier [contact], N. Revol [ARENAIRE Project].

MPFI is a library for multiprecision interval arithmetic, written in C (approximately 1000 lines), based on MPFR. It is developed in collaboration with N. Revol (ARENAIRE project). Initially, MPFI was developed for the needs of a new hybrid algorithm for the isolation of real roots of polynomials with rational coefficients. MPFI contains the same number of operations and functions as MPFR, the code is available and documented.

5.4. MPAI

Participants: P. Trébuchet [contact], F. Rouillier.

MPAI is a library for computing with algebraic infinitesimals. The infinitesimals are represented as truncated series. The library provides all the arithmetic functions needed to perform computations with infinitesimals. The interface is both GMP and RS compliant. It is implemented in the C language and represents approximately 1000 lines of code. The algorithms proposed in MPAI include Karatsuba's product and the short product, ...The code is available.

5.5. Interfaces

Participants: J.C. Faugère [contact], F. Rouillier.

In order to ease the use of the various software developed in the project, some conventions for the exchange of ASCII and binary files were developed and allow a flexible use of the servers GB, FGB or RS.

To make transparent the use of our servers from general computer algebra systems such as Maple or MuPAD we currently propose a common distribution for GB, FGB and RS including the servers as well as the interfaces for Maple and MuPAD. The instructions are illustrated by concrete examples and a simple installation process.

5.6. Gb

Participant: J.C. Faugère [contact].

Gb is one of the most powerful software for computing Gröbner bases currently diffused. Implemented in C/C++ (approximately 100000 lines), it is distributed since 1994 in the form of specific servers (direct computations, changes of orders, Hilbert's function, etc.). The initial interface (interactive system of commands) was

abandoned for lighter solutions (ASCII interface, UDX protocol) which are more powerful but for the moment more rudimentary. With the new algorithms proposed by J-C. Faugère ($F_n, n > 1$), GB is always maintained but is not developed any more. Indeed, data structures as well as basic algorithms necessary to implementations of these new methods being radically different from precedents, the GB servers will be gradually replaced by FGB servers. The existing prototypes of interfaces (ASCII, Maple, MuPAD, RS) were homogenized in order to provide a framework, initially based on GB, but evolutionary (towards FGB) in a transparent way. They will be kept. The future evolutions will follow then algorithmic progress.

5.7. FGb Servers

Participant: J.C. Faugère [contact].

As mentioned above, current implementations of the F_5 algorithm depend on many options. For efficiency reasons, it is currently preferable to compile specific servers, setting algorithms parameters like matrices sizes, linear algebra strategies for sparse matrices by hand. This has already successfully been done for path planning problems (parallel robots), and however helps to understand the main constraints in order to provide, in the future, software solutions that are independent from the type of system to be solved.

5.8. RS

Participant: F. Rouillier [contact].

RS is a software dedicated to the study of real roots of algebraic systems. It is entirely developed in C (100000 lines approximately) and succeeds to REALSOLVING developed during the European projects PoSSo and FRISCO. RS mainly contains functions for counting and isolating of real zeros of zero-dimensional systems. The user interfaces of RS are entirely compatible with those of GB/FGB (ASCII, MuPAD, Maple). RS is used in the project since several years and several development versions have been installed by numerous other teams. The following evolutions will depend on algorithmic progress and the users' needs (many internal functions are exported on demand).

5.9. Triangular Decompositions

Participant: P. Aubry [contact].

Triangular Decomposition is a library devoted to the decomposition of systems of polynomial equations and inequations and provides some tools for working with triangular sets. It decomposes the radical of the ideal generated by a family of polynomials into regular triangular sets that represents radical equidimensional ideals. It also performs the computation of polynomial gcd over an extension field or a product of such fields given by a triangular set.

A first version of this library is implemented in the Axiom computer algebra system. It is now developed in Magma.

5.10. RAGlib

Participant: M. Safey El Din [contact].

The **RAGlib** (**R**eal **A**lgebraic **G**eometry **L**ibrary) is a Maple library of symbolic algorithms devoted to some problems of Effective Real Algebraic Geometry, and more particularly, to the study of real solutions of polynomial systems of equations and inequalities. It contains algorithms performing:

- the *equi-dimensional decomposition* of an ideal generated by a polynomial family.
- the *emptiness test* of a real algebraic variety defined by a polynomial system of equations.
- the *computation of at least one point in each connected component* of a real algebraic variety defined by a polynomial system of equations.
- the *emptiness test* of a semi-algebraic set defined by a polynomial system of equations and non strict inequalities.
- the *computation of at least one point in each connected component* of a semi-algebraic set defined by a polynomial system of equations and non strict inequalities.

5.11. TCI

Participants: F. Rouillier [contact], J.C. Faugère.

As soon as they come from real applications, the polynomials resulting from processes of elimination (Gröbner bases, triangular sets) are very often too large to be studied by general computer algebra systems.

In the case of polynomials in two variables, a certified layout is enough in much cases to solve the studied problem (it was the case in particular for some applications in celestial mechanics). This type of layout is now possible thanks to the various tools developed in the project.

Two components are currently under development: the routine of computation (taking as input the polynomial function and returning a set of points) is stable (about 2000 lines in the C language, using the internal libraries of RS) and can be used as a black box in stand-alone mode or through Maple; the routine of layout is under study.

5.12. TSPR

Participants: F. Rouillier [contact], J.C. Faugère.

The purpose of project TSPR (Trajectory Simulator for Parallel Robots) is to homogenize and export the tools developed within the framework of our applications in parallel robotics. The software components of TSPR (about 1500 lines) are primarily written in C following the standards of RS and FGB and algorithms implemented in Maple using the prototypes of interfaces for FGB and RS . The encapsulation of all these components in a single distribution is available but not downloadable.

Prototypes of components for real-time resolution of certain types of systems now use also the ALIAS library developed in the INRIA project COPRIN.

5.13. NetTask

Participants: F. Rouillier [contact], J.C. Faugère.

We benefited during nearly one year of the assistance of Étienne Petitjean (SEDRE-LORIA) for this development.

The developed tool (about 11,000 lines in C++) makes possible the use of software installed on a network which then becomes usable even if it is not ready to be distributed and thus allows the experimental validation of algorithms proposed in articles or allows an external user to have a more precise idea of the possibilities offered by recent algorithmic progress. It will be useful in our project to export some of our prototypes.

NetTask is very flexible since it doesn't need any change in the software to be demonstrated. It is compatible with the current safety requirements and contains some powerful tools:

- allocation of the tasks launched by the users according to the availability of the software but also the load of the nodes on a given network of machines;
- system of queue for the tasks;
- complete control of the launched tasks by the user himself;
- many options of configuration such as for example the declaration of the machines and tasks available on a given network, the maximum number of tasks per user.

6. New Results

6.1. Interval Arithmetic

In [15], we justify why an arbitrary precision interval arithmetic is needed. To provide accurate results, interval computations require small input intervals; this explains why bisection is so often employed in interval algorithms. The MPFI library has been built in order to fulfill this need. Indeed, no existing library met the required specifications. The main features of this library are briefly given and a comparison with a fixed-precision interval arithmetic, on a specific problem, is presented. It shows that the overhead due to the multiple precision is completely acceptable. Eventually, some applications based on MPFI are given: robotics, isolation of polynomial real roots (by an algorithm combining symbolic and numerical computations) and approximation of real roots with arbitrary accuracy.

6.2. Arithmetic of $C_{3,4}$ curves

In [20], we provide explicit formulae for realising the group law in Jacobians of superelliptic curves of genus 3 and $C_{3,4}$ curves. It is shown that two distinct elements in the Jacobian of a $C_{3,4}$ curve can be added with 150 multiplications and 2 inversions in the field of definition of the curve, while an element can be doubled with 174 multiplications and 2 inversions. In superelliptic curves, 10 multiplications are saved.

6.3. Univariate polynomials

In [38] and [14], we explain how Bernstein's basis, widely used in Computer Aided Geometric Design, provides an efficient method for real root isolation, using De Casteljaeu's algorithm. We explain also the link between this approach and more classical methods for real root isolation [91]. Most of the content of the paper can be found in [74][45]. However, we present a new improved method for isolating real roots in Bernstein's basis inspired by [84].

6.4. Generalized Normal Forms

Let $K[x_1, \dots, x_n]$ be the ring of n -variate polynomials over the field K , and I an ideal of $K[x_1, \dots, x_n]$. Finding an effective representation of the quotient algebra $K[x_1, \dots, x_n]/I$ is a key point for studying the variety defined by I . Efficient ways to reach this goal exist, and the most used is to first compute a Gröbner basis, and next to derive an exact parameterization of the solution set. Though the work of Faugère made the Gröbner step rather efficient, the computation of Gröbner bases still suffers from very unstable behavior: the time and space needed for computing a Gröbner basis can vary greatly from a system to one another. Such changes obviously must appear when the structure of the variety defined by the system changes. However the behavior of the computation of Gröbner bases changes in a way uncorrelated to geometric changes of the variety. This is rather problematic as it implies that the actual cost of the solving process of a given system via Gröbner bases may be dramatically greater than what ought to be paid.

Our common work with Mourrain showed that in fact the methods of Macaulay and Buchberger share the same directing idea. Remark that Macaulay's method is known for its good stability properties, hence the idea to mix Macaulay's and Buchberger's approaches seems natural. The algorithm we provided so far was only shown to work for zero dimensional ideals. In the article [36] we provide a stopping criterion for this algorithm when the input system is positive dimensional. First we show that detecting that the ideal generated by the homogeneous parts of maximal degree of the polynomial sets constructed by the algorithm is k -regular (i.e. the Castelnuovo-Mumford regularity of this ideal is less than k) is enough to ensure that these sets allow to solve the *ideal membership* problem. Next, we show that detecting that this ideal is k -regular is simple using Quillen's theorem.

6.5. Complexity of Gröbner basis

In [19], we extend the notion of regular sequence (Macaulay) to overdetermined systems of algebraic equations. We study generic properties of Gröbner bases and analyze precisely the behavior of the F_5

algorithm. We extend the previous known results by defining the semi regular sequences and the degree of regularity for which we give a precise asymptotic analysis. For example as soon as the number m of polynomials is strictly greater than the number n of variables we gain a factor of 2 on the Macaulay bound and a factor 11.65 when $m = 2n$.

6.6. Gröbner bases over finite field

In [17], we compare the XL algorithm with the Gröbner basis algorithm. We explain the link between XL computation result and Gröbner basis with the well-known notion of D-Gröbner basis. Then we compare these algorithms in two cases: in the fields F_2 and F_q with $q < n$. For the field F_2 , we have proved that if XL needs to compute polynomials with degree D to terminate, the whole Gröbner basis is computed without exceeding that degree.

We have studied the XL algorithm and the F_5 algorithm on semi-regular sequences. We show that the size of matrices constructed by XL is huge compared to the ones of F_5 .

So the complexity of XL is worse than that of the F_5 algorithm on these systems. For the field F_q , we introduce an emulated algorithm using Gröbner basis computation to have a comparison between XL and Gröbner basis. We have proved that this algorithm will always reach a lower degree for intermediate polynomials than the XL algorithm. A study on semi-regular sequences shows that F_5 always has a better behavior than the XL algorithm especially when m is near from n .

6.7. Zero-dimensional Systems

In [31], we propose some new results about the computation and use of the Rational Univariate Representation. The first result is an algorithm that computes a RUR-candidate (no guarantee for the choice of a separating element), performing $O(\#\mathcal{J}D^{3/2} + nD^2)$ operations in \mathbb{Q} , taking as input \mathcal{J} , a multiplication table of $\mathbb{Q}[X_1, \dots, X_n]/I$ which is slightly better than [48]. The proposed algorithm is mainly the same as in [48] (based on the baby-step/giant-step algorithm [90]) but computes differently the so called "transposed product" (main operation). In addition, the output preserves the multiplicities. Basically, [48] generalizes the formulas from [81]: the coefficients of the RUR-candidate can directly be expressed with respect to $l(X_j t^i)$ where l is a "sufficiently generic" linear form. Taking for l the trace map (trace of the multiplication in $\mathbb{Q}[X_1, \dots, X_n]/I$), one recovers the formulas from [81]. The second result shows that one can check that the obtained RUR-candidate is a RUR performing $O(n\#\mathcal{J}D^{3/2})$ arithmetic operations, so that it produces the same output as [81] with the same computing time as the probabilistic computation of the RUR-candidate from [48]. We also propose an algorithm that computes all the sign conditions realized by a set of polynomials at the zeros of a zero-dimensional system. Its computing time complexity is D times less than the one of algorithm 11.18 p. 375 in [45] for the same input (\mathcal{J}). Given any polynomial f , the formulas from [81] or [48] can be used to compute a rational function $g_{t,f}/g_t$ which takes the same values as f at the zeroes of I . According to the results above, given a set of polynomials $\mathcal{F} = \{f_1, \dots, f_s\}$, one can compute a RUR and the rational functions g_{t,f_i} in $O(t_1 + sD^2)$ if t_1 is the computing time of the RUR. Thus, computing the sign conditions realized by the f_i leads to computing the sign conditions realized by the g_{t,f_i} at the roots of f_t . We can then substitute the "Sturm query" algorithm based on Hermite's quadratic form in [45] (computing time in $O(D^3)$ by its equivalent for univariate polynomials (see also for example [45]) - computing time in $O(D^2)$ for the naive version).

6.8. Parametric Polynomial Systems

In [37] and [27], we present a new algorithm for solving basic parametric constructible or semi-algebraic systems like $\mathcal{C} = \{x \in \mathbb{C}^n, p_1(x) = 0, \dots, p_s(x) = 0, f_1(x) \neq 0, \dots, f_l(x) \neq 0\}$ or $\mathcal{S} = \{x \in \mathbb{C}^n, p_1(x) = 0, \dots, p_s(x) = 0, f_1(x) > 0, \dots, f_l(x) > 0\}$, where $p_i, f_i \in \mathbb{Q}[U, X]$, $U = [U_1, \dots, U_d]$ is the set of parameters and $X = [X_{d+1}, \dots, X_n]$ the set of unknowns.

If Π_U denotes the canonical projection on the parameter's space, solving \mathcal{C} leads to compute sub-manifolds $\mathcal{U} \subset \overline{\Pi_U(\mathcal{C})}$ such that $(\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}, \Pi_U)$ is an analytic covering of \mathcal{U} (we say that \mathcal{U} has the (Π_U, \mathcal{C}) -covering

property). This guarantees that the cardinal of $\Pi_U^{-1}(u) \cap \mathcal{C}$ is locally constant on \mathcal{U} and that $\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}$ is a finite collection of sheets which are all locally homeomorphic to \mathcal{U} . In the case where $\Pi_U(\mathcal{C})$ is dense in \mathbb{C}^d , known algorithms for solving \mathcal{C} or \mathcal{S} ([52], [88], [94]) compute implicitly or explicitly a Zariski closed subset W such that any sub-manifold of $\overline{\Pi_U(\mathcal{C})} \setminus W$ has the (Π_U, \mathcal{C}) -covering property.

We introduce the *discriminant varieties of \mathcal{C} with respect to Π_U* which are algebraic sets with the above property. We then show that the set of points of $\overline{\Pi_U(\mathcal{C})}$ which do not have any neighborhood with the (Π_U, \mathcal{C}) -covering property is a Zariski closed set and thus define the *minimal discriminant variety of \mathcal{C} with respect to Π_U* , and we propose an algorithm to compute it efficiently. Thus, solving \mathcal{C} (resp. \mathcal{S}) remains to describing $\overline{\Pi_U(\mathcal{C})} \setminus W_D$ (resp. $(\overline{\Pi_U(\mathcal{C})} \setminus W_D) \cap \mathbb{R}^d$) which can be done using critical point methods such as in [45] or partial CAD based strategies [52].

6.9. Parametric varieties and rational maps

Our work on parametric varieties and rational maps began with an application of our methods to statistics which leads us to a complete study of the topology of the image of a rational mapping [12]. This application became our first significant example of a parametric polynomial system with a non dense projection on the space of the parameters, and was therefore a fundamental tool for extending correctly our theory of discriminant varieties to this case.

Rational maps appear in many application domains, whenever some measurable quantities are rational functions of some hidden state variables. They appear also in the definition of the parametric varieties, which are widely used in geometrical design. On the other hand, their linearity with respect to their image leads to shortcuts in many computations, especially the one of the discriminant variety.

Therefore their study deserves a special effort which started with [12], has been continued with [24],[37] and will be submitted to the book following the AGGM conference.

6.10. Positive dimensional systems of equations

The results of [16] have been published. In this article, we provide an algorithm computing at least one point in each connected component of a real algebraic set defined by a polynomial system of equations. The used strategy is based on the computation of critical loci of some projections and their respective sets of non-properness. The paper ends with intensive experimental results. These results have been improved in [85].

In [39], we have provided some sharp bounds on the number of critical points of a polynomial mapping restricted to a smooth algebraic variety. These bounds have been obtained by exploiting the bi-homogeneous structure of the Lagrange system and proving a Bezout bound to such systems on the sum of the degree of the isolated primary components of the ideal generated by the studied polynomial system. Then, we have used these results to improve the Thom-Milnor bound which majorates the maximal number of connected components of a real algebraic set. This is done by generalizing the algorithm of [85] to non equidimensional situations for which some optimal bounds on the degree of the output and the number of arithmetic operations. These results have been submitted to Journal of Complexity.

6.11. Generalized critical values and semi-algebraic systems

In [28], we provide an algorithm computing at least one point in each connected component of a semi-algebraic set defined by a system of equations and non-strict inequalities. We prove improved complexity bounds to tackle this problem, provide an implementation, and study an application to pattern-matching problems.

In [35], we focus on the case of semi-algebraic sets defined by a single polynomial inequality or a single polynomial inequation. In this case, a classical strategy consists in introducing an infinitesimal and performing computations on a Puiseux series field. We substitute this strategy by finding an *a priori* good specialization value for this infinitesimal. This is done by computing the set of generalized critical values of a polynomial mapping. The complexity of the obtained algorithm is better than the previous ones and an implementation

is provided. The extended abstract presenting these results has been accepted for an oral communication at ICPSS'04 (International Conference on Polynomial System Solving) which has been held at Pierre et Marie Curie University, Paris. The full paper is in preparation and will be submitted to the Special Issue of Journal of Symbolic Computation in honour of Daniel Lazard.

7. Contracts and Grants with Industry

7.1. CELAR (DGA)

Participants: G. Ars, J.C. Faugère [contact].

The objective is to evaluate, on examples of realistic size, attacks of algebraic type (and in particular Gröbner bases) on filtered registers (stream ciphers). This kind of crypto-systems is, in particular, used in mobile communications (mobile phones, decoders...) because the needed resources are easy to implement in hardware and allows extremely fast encoding/decoding. The workplan proposed by this contract is original since Celar emits "challenges" of realistic size (that means registers of size from 60 to 128 bits); the attack step induces a specific modelization of the problem (which represents a significant part of the computation time) then a resolution step which uses our most efficient algorithms (F5 modulo 2). It should be pointed out that all the results of this study are publishable and that this contract is in fact a true scientific partnership.

7.2. Sciface Software

Participants: F. Rouillier [contact], P. Zimmermann [SPACES project].

In November 2001, a cooperation agreement around the UDXF program for binary data exchange was signed between the SciFace company (which distributes the computer algebra system MuPAD) and INRIA Lorraine (representing the University Pierre and Marie Curie). UDXF takes as parameter a communication protocol, for example that of patent UDX, but it can also be another protocol. Within the framework of this agreement, SciFace has the right to use UDXF for the MuPAD-Scilab interface. In exchange, SciFace takes part in the development and the improvement of UDX. Version 2.5.x of MuPAD, distributed since September 2002, integrates this interface.

7.3. Thales

Participants: J.C. Faugère [contact], S. Lacharte.

The objective is to evaluate, on examples of realistic size, attacks of algebraic type (and in particular Gröbner bases) on filtered registers (stream ciphers). This kind of crypto-systems is, in particular, used in mobile communications (mobile phones, decoders...) because the needed resources are easy to implement in hardware and allows extremely fast encoding/decoding. The workplan proposed by this contract is original since Celar emits "challenges" of realistic size (that means registers of size from 60 to 128 bits); the attack step induces a specific modelization of the problem (which represents a significant part of the computation time) then a resolution step which uses our most efficient algorithms (F5 modulo 2). It should be pointed out that all the results of this study are publishable and that this contract is in fact a true scientific partnership.

7.4. National Actions

7.4.1. Ministry Grant (ACI) "Cryptologie"

Participants: J.C. Faugère [contact for SALSA], G. Hanrot [SPACES project - head].

This project started in 2002 and will end in 2004.

The main goal of this project is to study the interactions between cryptography and Computer Algebra and more specifically the impact of fast polynomial system algorithms on public key cryptosystems based on

multivariate (quadratic) equations (such as HFE). For instance a member of this project was able to “break” the first HFE challenge by computing a huge Gröbner basis.

7.5. European Actions

7.5.1. *European Network RAAG*

Participants: F. Rouillier [contact], M. Safey El Din.

This project started in 2002 and will end in 2006.

The SPACES project takes part in the European project RAAG (Real Algebraic and Analytic Geometry), F. Rouillier being involved in the applications and connection with industry item.

RAAG is a Research Training Network of the "Human Potential" program of the European Commission, sponsored for 48 months starting from March 2002. The network is managed by the University of Passau (Germany), the coordination of the French team being ensured by the team of real geometry and computer algebra of the University of Rennes I.

The main goal of this project is to increase the links between the various fields of research listed in the topics of the project (real algebraic geometry, analytical geometry, complexity, formal calculation, applications, etc.) by means of conferences, schools and exchanges of young researchers. It brings together a great number of European teams and in particular the majority of the French teams working in the scientific fields falling under the topics of the project.

8. Dissemination

8.1. Scientific Animation

8.1.1. *ICPSS*

The members of the project organized jointly with the CALFOR team the first International Conference on Polynomial System Solving. This first edition was in honor of Daniel Lazard. About 100 participants from various countries were present and prestigious researchers gave invited lectures (B. Buchberger, J. Calmet, J. Davenport, H. Hong and M.-F. Roy). A special issue of Journal of Computation is planned for 2005.

8.1.2. *Invited lectures*

- P. Aubry - Bejin University [18];
- J.-C. Faugère - AES IV [23];
- D. Lazard - ECCAD [25];
- F. Rouillier - Dagstuhl [32] - Berkeley (MSRI)[29] - Bejin University [33];

9. Bibliography

Major publications by the team in recent years

- [1] P. AUBRY, D. LAZARD, M. M. MAZA. *On the theories of triangular sets*, in "Journal of Symbolic Computation", vol. 28, 1999, p. 105-124.
- [2] P. AUBRY, F. ROUILLIER, M. SAFEY. *Real Solving for Positive Dimensional Systems*, in "Journal of Symbolic Computation", vol. 34, n° 6, 2002, p. 543-560.
- [3] J.-C. FAUGÈRE. *A new efficient algorithm for computing Gröbner bases without reduction to zero F_5* , in "International Symposium on Symbolic and Algebraic Computation Symposium - ISSAC 2002, Villeneuve d'Ascq, France", Jul 2002.
- [4] J.-C. FAUGÈRE. *A New Efficient Algorithm for Computing Gröbner bases (F_4)*, in "Journal of Pure and Applied Algebra", vol. 139, n° 1-3, June 1999, p. 61-88.
- [5] J.-C. FAUGÈRE, A. JOUX. *Algebraic Cryptanalysis of Hidden Field Equation (HFE) Cryptosystems Using Gröbner Bases*, in "CRYPTO 2003", 2003, p. 44-60.
- [6] J.-C. FAUGÈRE, F. M. DE SAINT MARTIN, F. ROUILLIER. *Design of regular nonseparable bidimensional wavelets using Groebner basis techniques*, in "IEEE SP Transactions Special Issue on Theory and Applications of Filter Banks and Wavelets", vol. 46, n° 4, apr 1998, p. 845-856.
- [7] F. ROUILLIER. *Solving zero-dimensional systems through the rational univariate representation*, in "Journal of Applicable Algebra in Engineering, Communication and Computing", vol. 9, n° 5, 1999, p. 433-461.
- [8] F. ROUILLIER, P. ZIMMERMANN. *Efficient Isolation of Polynomial Real Roots*, in "Journal of Computational and Applied Mathematics", vol. 162, n° 1, 2003, p. 33-50.
- [9] M. SAFEY EL DIN, E. SCHOST. *Polar varieties and computation of one point in each connected component of a smooth real algebraic set*, in "International Symposium on Symbolic and Algebraic Computation 2003 - ISSAC'2003, Philadelphia, USA", J. SENDRA (editor), ACM Press, aug 2003, p. 224-231.

Doctoral dissertations and Habilitation theses

- [10] M. BARDET. *On the complexity of Gröbner basis computation of semi-regular overdetermined algebraic equations*, Ph. D. Thesis, Université Pierre et Marie Curie (Paris VI), 2004.

Articles in referred journals and book chapters

- [11] A. BASIRI, A. ENGE, J.-C. FAUGÈRE, N. GUREL. *The arithmetic of Jacobian groups of superelliptic cubics*, in "Mathematics of Computation", 2004.
- [12] D. LAZARD. *Injectivity of real rational mappings: the case of a mixture of two Gaussian laws*, in "Math. Comput. Simulation", vol. 67, n° 1-2, 2004, p. 67-84.

- [13] D. LAZARD. *Solving quintics by radicals*, in "The legacy of Niels Henrik Abel, Berlin", Springer, 2004, p. 207–225.
- [14] B. MOURRAIN, F. ROUILLIER, M.-F. ROY. *Bernstein's basis and real root isolation*, in "Current trends in combinatorial and computational geometry", vol. Special volume devoted to special program. "Discrete and Computational Geometry" at MSRI in its series "Mathematical Sciences Research Institute Publications", Cambridge University Press, to appear.
- [15] N. REVOL, F. ROUILLIER. *Motivations for an arbitrary precision interval arithmetic and the MPFI library*, in "Reliable Computing", to appear.
- [16] M. SAFEY EL DIN, E. SCHOST. *Properness defects of projection functions and computation of at least one point in each connected component of a real algebraic set*, in "Journal of Discrete and Computational Geometry", sep 2004.

Publications in Conferences and Workshops

- [17] G. ARS, J. FAUGÈRE. *Comparison of XL and Gröbner basis algorithms over Finite Fields*, in "AsiaCrypt 2004", 2004.
- [18] P. AUBRY. *Triangular Sets in Real Solving*, in "Invited Lecture - The Eighth International Workshop on Mathematics Mechanization, Peking University, Beijing, China", <http://www.mmrc.iss.ac.cn/~iwmm/>, Jul 2004.
- [19] M. BARDET, J. FAUGÈRE, B. SALVY. *On the complexity of Gröbner basis computation of semi-regular overdetermined algebraic equations*, in "Proceedings of the International Conference on Polynomial System Solving", 2004, p. 71–74.
- [20] A. BASIRI, A. ENGE, J. FAUGÈRE, N. GÜREL. *Implementing the arithmetic of $c34$ curves.*, in "Sixth Algorithmic Number Theory Symposium, ANTS VI proceedings (Vermont, USA, June 13–18, 2004)", D. BUELL (editor), LNCS, vol. 3076, Springer-Verlag, 2004, p. 57–71.
- [21] L. DUPONT, S. LAZARD, D. LAZARD, S. PETITJEAN. *Near-Optimal Parameterization of the Intersection of Quadrics: Theory and Implementation*, in "International Conference on Polynomial System Solving", 2004, p. 57–59.
- [22] J. FAUGÈRE. *"Real Time" Gröbner basis computation*, in "Journées LNF - Toulouse", 2004.
- [23] J. FAUGÈRE. *Fast Gröbner. Algebraic cryptanalysis of HFE and Filter Generators*, in "Invited lecture - AES 2004 - Bonn", 2004.
- [24] D. LAZARD. *Grobner bases for implicitisation, local inversibility and related questions*, in "AGGM (Algebraic Geometry and Geometric Modeling)", 2004.
- [25] D. LAZARD. *Solving zero-dimensional systems of equations and inequations, depending on parameters*, in "ECCAD (East Coast Computer Algebra Day), Waterloo, Canada", 2004, <http://www.cargo.wlu.ca/ECCAD2004/>.

- [26] D. LAZARD. *Testing membership of an approximate point to a parametrized variety*, in "Journées LNF - Toulouse", 2004.
- [27] D. LAZARD, F. ROUILLIER. *Solving Parametric Polynomial Systems*, in "International Conference on Polynomial System Solving", 2004, p. 53–55.
- [28] C. LE GUERNIC, F. ROUILLIER, M. SAFEY EL DIN. *Computing sampling points in a semi-algebraic set defined by non-strict inequalities, Application to Pattern-Matching Problems*, in "EACA, Santander", ACM Press, 2004.
- [29] F. ROUILLIER. *Exact Computations and Real Roots of Polynomial Systems*, in "Invited Lecture - Workshop on Algorithmic, Combinatorial and Applicable Real Algebraic Geometry, MSRI - Berkeley", video and overheads on <http://www.msri.org/publications/ln/msri/2004/acarag/rouillier/1/>, 2004.
- [30] F. ROUILLIER. *Gröbner based Real Solving*, in "Journées LNF - Toulouse", 2004.
- [31] F. ROUILLIER. *On the Rational Univariate Representation*, in "International Conference on Polynomial System Solving", 2004, p. 75–79.
- [32] F. ROUILLIER. *Real Roots of parametric Systems of polynomial equalities and inequalities*, in "Workshop on Real Computation and Complexity", Dagstuhl, Germany, 2004.
- [33] F. ROUILLIER. *Real roots of parametric polynomial systems.*, in "Invited Lecture - The Eighth International Workshop on Mathematics Mechanization, Peking University, Beijing, China", Jul 2004, <http://www.mmrc.iss.ac.cn/~iwmm/>.
- [34] M. SAFEY EL DIN. *Generalized critical values and solving polynomial inequalities*, in "Journées LNF - Toulouse", 2004.
- [35] M. SAFEY EL DIN. *Generalized critical values and solving polynomial inequalities*, in "Proceedings of the International Conference on Polynomial System Solving", Nov 2004, p. 40–42.
- [36] P. TREBUCHET. *Generalized normal forms for positive dimensional ideals*, in "ICPSS 2004", 2004.

Internal Reports

- [37] D. LAZARD, F. ROUILLIER. *Solving Parametric Polynomial Systems*, Technical report, n° RR-5322, INRIA, Oct 2004, <http://www.inria.fr/rrrt/rr-5322.html>.
- [38] B. MOURRAIN, F. ROUILLIER, M.-F. ROY. *Bernstein's basis and real root isolation*, Technical report, n° RR-5149, INRIA, 2004, <http://www.inria.fr/rrrt/rr-5149.html>.
- [39] M. SAFEY EL DIN, P. TRÉBUCHET. *Strong bihomogeneous Bézout theorem and degree bounds for algebraic optimization*, submitted to Journal of Pure and Applied Algebra, Technical report, n° 5071, INRIA, 2004, <http://www.inria.fr/rrrt/rr-5071.html>.

Bibliography in notes

- [40] P. AUBRY. *Ensembles triangulaires de polynômes et résolution de systèmes algébriques*, Ph. D. Thesis, Université Paris 6, France, 1999.
- [41] P. AUBRY, D. LAZARD, M. M. MAZA. *On the theories of triangular sets*, in "Journal of Symbolic Computation", vol. 28, 1999, p. 105-124.
- [42] P. AUBRY, F. ROUILLIER, M. SAFEY. *Real Solving for Positive Dimensional Systems*, in "Journal of Symbolic Computation", vol. 34, n° 6, 2002, p. 543–560.
- [43] P. AUBRY, A. VALIBOUZE. *Using Galois ideals for computing relative resolvents*, in "J. of Symbolic Computation", Special Issue on Algorithmic Galois Theory, vol. 30, n° 6, 2000, p. 635-651.
- [44] A. BASIRI, J.-C. FAUGÈRE. *Changing the ordering of Gröbner bases with LLL: case of two variables*, in "ISSAC 2003", 2003, p. 23-29.
- [45] S. BASU, R. POLLACK, M.-F. ROY. *Algorithms in real algebraic geometry*, Algorithms and Computations in Mathematics, vol. 10, Springer-Verlag, 2003.
- [46] S. BASU, R. POLLACK, M. ROY. *On the combinatorial and algebraic complexity of quantifier elimination*, in "Journal of Assoc. Comput. Machin.", 1996, p. 1002–1045.
- [47] S. BASU, R. POLLACK, M.-F. ROY. *A new algorithm to find a point in every cell defined by a family of polynomials*, in "Quantifier elimination and cylindrical algebraic decomposition", Springer-Verlag, 1998.
- [48] A. BOSTAN, B. SALVY, É. SHOST. *Fast Algorithms for Zero-Dimensional Polynomial Systems Using Duality*, in "Applicable Algebra in Engineering, Communication and Computing", vol. 14, n° 4, 2003, p. 239 - 272.
- [49] B. BUCHBERGER. "*Groebner bases : an algorithmic method in polynomial ideal theory*", Recent trends in multidimensional systems theory, Reider ed. Bose, 1985.
- [50] B. BUCHBERGER, G.-E. COLLINS, R. LOOS. *Computer Algebra Symbolic and Algebraic Computation*, second edition, Springer-Verlag, 1982.
- [51] G. COLLINS. *Quantifier elimination for real closed fields by cylindrical algebraic decomposition*, in "Springer Lecture Notes in Computer Science 33", vol. 33, 1975, p. 515-532.
- [52] G. E. COLLINS, H. HONG. *Partial Cylindrical Algebraic Decomposition*, in "Journal of Symbolic Computation", vol. 12, n° 3, 1991, p. 299–328.
- [53] J. FAUGÈRE, F. M. DE SAINT-MARTIN, F. ROUILLIER. *Une famille de bancs de filtres 2D non séparables*, 1997, Patent.
- [54] J.-C. FAUGÈRE. *A new efficient algorithm for computing Gröbner bases without reduction to zero F_5* , in

"International Symposium on Symbolic and Algebraic Computation Symposium - ISSAC 2002, Villeneuve d'Ascq, France", Jul 2002.

- [55] J.-C. FAUGÈRE. *A New Efficient Algorithm for Computing Gröbner bases (F_4)*, in "Journal of Pure and Applied Algebra", vol. 139, n° 1-3, June 1999, p. 61-88.
- [56] J. FAUGÈRE, P. GIANNI, D. LAZARD, T. MORA. *Efficient Computation of Zero-Dimensional Gröbner Basis by Change of Ordering*, in "Journal of Symbolic Computation", vol. 16, n° 4, Oct. 1993, p. 329–344.
- [57] J. FAUGÈRE, F. ROUILLIER. *Dispositif d'échanges de données entre matériels informatiques*, 1999, Patent proposal number 99 08172.
- [58] J.-C. FAUGÈRE, F. ROUILLIER. *Design of filter and filter banks using dedicated Computer Algebra Tools*, in "International Conference on Applications of Computer Algebra (ACA'99)", Applications of Computer Algebra to Signal Processing, Jeremy Johnson and Markus Pueschel, 1999.
- [59] J.-C. FAUGÈRE, F. M. DE SAINT MARTIN, F. ROUILLIER. *Design of regular nonseparable bidimensional wavelets using Groebner basis techniques*, in "IEEE SP Transactions Special Issue on Theory and Applications of Filter Banks and Wavelets", vol. 46, n° 4, Apr 1998, p. 845-856.
- [60] J. FAUGÈRE, D. LAZARD. *The Combinatorial Classes of Parallel Manipulators*, in "Mechanism and Machine Theory", vol. 30, 1995, p. 765–776.
- [61] D. GRIGOR'EV, N. VOROBJOV. *Solving Systems of Polynomial Inequalities in Subexponential Time*, in "J. Symbolic Comput.", vol. 5, 1988, p. 37–64.
- [62] J. HEINTZ, M.-F. ROY, P. SOLERNÓ. *On the Complexity of Semi-Algebraic Sets*, in "Proc. IFIP 89, San Francisco", 1989, p. 293-298.
- [63] J. HEINTZ, M.-F. ROY, P. SOLERNO. *On the Theoretical an Practical Complexity of the Existential Theory of Reals*, in "The Computer Journal", vol. 36, n° 5, 1993, p. 427-431.
- [64] A. J. J.-C. FAUGÈRE. *Algebraic Cryptanalysis of Hidden Field Equation (HFE) Cryptosystems Using Gröbner Bases*, in "CRYPTO 2003", 2003, p. 44-60.
- [65] M. KALKBRENNER. *Three contributions to elimination theory*, Ph. D. Thesis, Johannes Kepler University, Linz, 1991.
- [66] D. LAZARD. *Resolution of polynomial systems*, in "4th Asian Symposium on Computer Mathematics - ASCM 2000, Chiang Mai, Thailand", Lecture Notes Series on Computing, vol. 8, World Scientific, Dec 2000, p. 1 - 8.
- [67] D. LAZARD. *On the specification for solvers of polynomial systems*, in "5th Asian Symposium on Computers Mathematics -ASCM 2001", Lecture Notes Series in Computing, vol. 9, World Scientific, 2001, p. 66-75.

-
- [68] D. LAZARD. *A new method for solving algebraic systems of positive dimension*, in "Discrete Applied Mathematics", 1991.
- [69] D. LAZARD. *Solving Zero - dimensional algebraic systems*, in "Journal of Symbolic Computation", vol. 13, 1992, p. 117-132.
- [70] D. LAZARD. *Stewart platforms and Gröbner basis*, in "Proceedings of Advances in Robotics Kinematics", Sep 1992, p. 136-142.
- [71] D. LAZARD, J.-P. MERLET. *The (true) Stewart platform has 12 configurations*, in "Proc. of IEEE Conference on Robotics and Vision, San Diego", 1994.
- [72] J. LEBRUN, I.-W. . SELESNICK. *Gröbner bases and wavelet design*, in "Journal of Symbolic Computation", vol. 37, n° 2, 2004, p. 227-259.
- [73] H. LOMBARDI, M.-F. ROY, M. S. E. DIN. *New Structure Theorems for subresultants*, in "Journal of Symbolic Computations", May, 2000.
- [74] B. MOURRAIN, M. VRAHATIS, J. YAKOUBSOHN. *On the Complexity of Isolating Real Roots and Computing with Certainty the Topological Degree*, in "Journal of Complexity", vol. 18, n° 2, 2002, p. 612–640.
- [75] J. E. OMRI. *Analyse géométrique et cinématique des mécanismes de type manipulateur*, Ph. D. Thesis, Université de Nantes, Feb. 1996.
- [76] J. RITT. *Differential equations from an algebraic standpoint*, vol. 14, 1932.
- [77] F. ROUILLIER. *Efficient algorithms based on critical points method*, in "Algorithmic and Quantitative Real Algebraic Geometry", S. BASU, L. GONZALEZ-VEGA (editors)., DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 60, American Mathematical Society, 2003, p. 123–138.
- [78] F. ROUILLIER. *Efficient real solutions and robotics*, in "First EMS -SMAI -SMF Joint Conference Applied Mathematics and Applications of Mathematics", 2003.
- [79] F. ROUILLIER. *Real Root Counting For some Robotics problems*, in "Solid Mechanics and its Applications, Kluwer Academic Publishers", vol. 40, 1995, p. 73-82.
- [80] F. ROUILLIER. *Algorithmes efficaces pour l'étude des zéros réels des systèmes polynomiaux*, Ph. D. Thesis, Université de Rennes I, may 1996.
- [81] F. ROUILLIER. *Solving zero-dimensional systems through the rational univariate representation*, in "Journal of Applicable Algebra in Engineering, Communication and Computing", vol. 9, n° 5, 1999, p. 433–461.
- [82] F. ROUILLIER, M. ROY, M. SAFEY. *Finding at least one point in each connected component of a real algebraic set defined by a single equation*, in "Journal of Complexity", vol. 16, 2000, p. 716–750.
- [83] F. ROUILLIER, M. SAFEY, É. SCHOST. *Solving the Birkhoff Interpolation Problem via the Critical Point*

Method: An Experimental Study, in "Automated Deduction in Geometry - Third International Workshop ADG 2000, Zurich Switzerland, September 2000, Revised Papers", J. RICHTER-GEBERT, D. WANG (editors), Lecture Notes in Artificial Intelligence, n° 2061, Springer, 2001, p. 26–40.

- [84] F. ROUILLIER, P. ZIMMERMANN. *Efficient Isolation of Polynomial Real Roots*, in "Journal of Computational and Applied Mathematics", vol. 162, n° 1, 2003, p. 33-50.
- [85] M. SAFEY EL DIN, E. SCHOST. *Polar varieties and computation of one point in each connected component of a smooth real algebraic set*, in "International Symposium on Symbolic and Algebraic Computation 2003 - ISSAC'2003, Philadelphia, USA", J. SENDRA (editor), ACM Press, aug 2003, p. 224-231.
- [86] M. SAFEY EL DIN. *RAGLib (Real Algebraic Library Maple package)*, 2003, <http://www-calfor.lip6.fr/~safey/RAGLib>.
- [87] M. SAFEY EL DIN. *Résolution Réelle des Systèmes Polynomiaux en Dimension Positive*, Ph. D. Thesis, Université de Paris VI, 2001.
- [88] É. SCHOST. *Sur la résolution des systèmes polynomiaux à paramètres*, Ph. D. Thesis, École polytechnique, 2000.
- [89] É. SCHOST. *Computing Parametric Geometric Resolutions*, in "Applicable Algebra in Engineering, Communication and Computing", vol. 13, n° 5, 2003, p. 349 - 393.
- [90] V. SHOUP. *Efficient Computation of Minimal Polynomials in Algebraic Extensions of Finite Fields*, in "ISSAC'99".
- [91] J. USPENSKY. *Theory of equations*, MacGraw Hill, 1948.
- [92] V. WEISPFENNING. *Canonical comprehensive Gröbner bases*, in "Proceedings of the 2002 international symposium on Symbolic and algebraic computation", ACM Press, 2002, p. 270–276.
- [93] V. WEISPFENNING. *Comprehensive Gröbner bases*, in "Journal of Symbolic Computation", vol. 14, 1992, p. 1–29.
- [94] V. WEISPFENNING. *Solving parametric polynomial equations and inequalities by symbolic algorithms*, World Scientific, 1995.
- [95] P. WENGER, J. E. OMRI. *Changing Posture for cuspidal Robot Manipulators*, in "Proceeding of the 1996 IEEE Int. Conf on Robotics and Automation", 1996, p. 3173-3178.
- [96] P. WENGER. *Some guidelines for the kinematic design of new manipulators*, in "Mechanism and Machine Theory", vol. 35, 2000, p. 437-449.
- [97] P. WENGER. *Classification of 3R Positioning Manipulators*, in "Journal of Mechanical Design", vol. 120, June 1998, p. 327-332.

-
- [98] W. T. WU. *Basic principles of mechanical theorem proving in elementary geometries*, in "J. Sys. Sci. Math. Sci.", vol. 4, 1984, p. 207–235.
- [99] W. T. WU. *A Zero Structure Theorem for polynomial equations solving*, in "MM Research Preprints", vol. 1, 1987, p. 2-12.
- [100] L. YANG, J. ZHANG. *Searching dependency between algebraic equations: an algorithm applied to automated reasoning*, in "Artificial intelligence in mathematics", JOHNSON/MCKEE/VELLA (editor), Oxford University Press, 1994, p. 147–156.