INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Project-Team Café

Calcul Formel et Équations

Sophia Antipolis

THEME SYM

Activity Report

2005

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2. Overall Objectives

2.1. Overall Objectives

Our goal is to develop computer algebra methods and software for solving functional equations, i.e. equations where the unknowns represent functions rather than numerical values, as well as to foster the use of such methods in engineering by producing the programs and tools necessary to apply them to industrial problems. We study in particular linear and nonlinear differential and (q)-difference equations, partial and ordinary.

3. Scientific Foundations

3.1. Differential ideals and D-modules

Keywords: *D-modules*, *algebraic analysis*, *control theory*, *differential algebra*, *differential elimination*, *differential systems*, *formal integrability*, *formal integrability*, *holonomic systems*, *involution*.

Algorithms based on algebraic theories are developed to investigate the structure of the solution set of general differential systems. Different algebraic and geometric theories are the sources of our algorithms.

3.1.1. Differential elimination and completion

Formal integrability is the first problem that our algorithms address. The idea is to *complete* a system of partial differential equation so as to be in a position to compute the *Hilbert differential dimension polynomial* or equivalently, its coefficients, the *Cartan characters*. Those provide an accurate measure of the arbitrariness that comes in the solution set (how many arbitrary functions of so many variables). Closely related is the problem of determining the *initial conditions* that can be freely chosen for having a well-posed problem (i.e. that lead to existence and uniqueness of solutions). This is possible if we can compute all the differential relations up to a given order, meaning that we cannot obtain equations of lower order by combining the existing equations in the system. Such a system is called *formally integrable* and numerous algorithms for making systems of partial differential equations formally integrable have been developed using different approaches by E. Cartan [25], C. Riquier [61], M. Janet [44] and D. Spencer [67].

Differential elimination is the second problem that our algorithms deal with. One typically wants to determine what are the lowest differential equations that vanish on the solution set of a given differential system. The sense in which *lowest* has to be understood is to be specified. It can first be order-wise, as it is of use in the formal integrability problem. But one can also wish to find differential equations in a subset of the variables, allowing the model to be reduced.

The radical differential ideal generated by a set of differential polynomials $\mathcal{S}$, i.e. the left hand side of differential equations where the right hand side is zero, is the largest set of differential polynomials that vanish on the solution set of $\mathcal{S}$. This is the object that our algorithms manipulate and for which we compute adequate representations in order to answer the above questions.

In the nonlinear case the best we can hope for is to have information outside of some hypersurface. Actually, the radical differential differential ideal can be decomposed into components on which the answers to formal integrability and eliminations are different. For each component the *characteristic set* delivers the information about the singular hypersurface together with the quasi-generating set and membership test.

Triangulation-decomposition algorithms perform the task of computing a characteristic set for all the components of the radical differential ideal generated by a finite set of differential polynomials. References for those algorithms are the book chapters written by E. Hubert [5], [4]. They are based on the differential algebra developed by Ritt [62] and Kolchin [46].

The objectives for future research in the branch of triangulation-decomposition is the improvement of the algorithms, the development of alternative approaches to certain class of differential systems and the study of the intrinsic complexity of differential systems.

Another problem, specific to the nonlinear case, is the understanding and algorithmic classification of the different behaviors of interference of the locus of one component on the locus of another. The problem becomes clear in the specific case of radical differential ideals generated by a single differential polynomial. One then wishes to understand the behavior of non singular solutions in the vicinity of singular solutions. Only the case of the first order differential polynomial equations is clear. Singular solutions are either the envelope or the limit case of the non singular solutions and the classification is algorithmic [62], [43].

3.1.2. Algebraic analysis

When the set $\mathcal{S}$ represent linear differential equations, we use the theory of D-modules (algebraic analysis, based on algebraic techniques such as module theory and homological algebra) as well as the formal integrability theories mentioned in the previous section (based on geometric techniques such as differential manifolds, jet spaces, involution and Lie groups of transformations). Using the duality between matrices of differential operators and differential modules, we can apply techniques that have been developed independently for those two approaches.

In addition, thanks to the works of B. Malgrange [49], V. Palamodov [58] and M. Kashiwara [45], the theory of D-modules yields new results and information about the algebraic and analytic properties of systems of linear partial differential equations, their solutions and associated geometric invariants (e.g. characteristic varieties). The above theories are becoming algorithmic thanks to the recent development of Gröbner bases [28] and involutive bases in rings of differential operators, enabling the implementation of efficient algorithms for making systems formally integrable, as well as for computing special closed-form solutions [53], [54]. By using formal adjoints, it is now finally possible to algebraically study systems of linear partial differential equations, and our objectives in that field are: (i) to develop and implement efficient algorithms for computing the polynomial and rational solutions of such systems and, further, for factoring and decomposing their associated D-modules; (ii) to study the links between the algebraic and analytic properties of such systems (since the algorithmic determination of the algebraic properties yields information about the analytic properties); (iii) to apply the above algorithms to the design and analysis of linear control systems.

3.2. Groups of transformations

Keywords: *Hamiltonian mechanics*, *closed-form solutions*, *differential Galois group*, *differential invariants*, *dynamical systems*, *formal integrability*, *linear systems of partial differential equations*, *nonlinear differential systems*, *symmetry*, *variational equations*.

3.2.1. Lie groups of transformations

Though not a major subject of expertise, the topic is at the crossroads of the algorithmic themes developed in the team.

The Lie group, or symmetry group, of a differential system is the (biggest) group of point transformations leaving the solution set invariant. Besides the group structure, a Lie group has the structure of a differentiable manifold. This double structure allows to concentrate on studying the tangent space at the origin, the Lie algebra. The Lie group and the Lie algebra thus capture the geometry of a differential system. This geometric knowledge is exploited to *solve* nonlinear differential systems.

The Lie algebra is described by the solution set of a system of linear partial differential equations, whose determination is algorithmic. The dimension of the solution space of that linear differential system is the dimension of the Lie group and can be determined by the tools described in Section 3.1. Explicit subalgebras can be determined thanks to the methods developed within the context of Section 3.2.2.

For a given group of transformations on a set of independent and dependent variables there exist invariant derivations and a finite set of differential invariants that generate all the differential invariants [69]. This forms an intrinsic frame for expressing any differential system invariant under this group action. This line of ideas took a pragmatic shape for computation with the general method of M. Fels and P. Olver [34] for computing the generating set of invariants. The differential algebra they consider has features that go beyond classical differential algebra. We are engaged in investigating the algebraic and algorithmic aspect of the subject.

3.2.2. Galois groups of linear functional equations

Differential Galois theory, developed first by Picard and Vessiot, then algebraically by Kolchin, associates a linear algebraic group to a linear ordinary differential equation or system. Many properties of its solutions, in particular the existence of closed-form solutions, are then equivalent to group-theoretic properties of the associated Galois group [65]. By developing algorithms that, given a differential equation, test such properties, Kovacic [47] and Singer [64] have made the existence of closed-form solutions decidable in the case of equations with polynomial coefficients. Furthermore, a generalization of differential Galois theory to linear ordinary difference equations [66] has yielded an algorithm for computing their closed-form solutions [40]. Those algorithms are however difficult to apply in practice (except for equations of order two) so many algorithmic improvements have been published in the past 20 years. Our objectives in this field are to improve the efficiency of the basic algorithms and to produce complete implementations, as well as to generalize them and their building blocks to linear partial differential and difference equations.

An exciting application of differential Galois theory to dynamical systems is the Morales-Ramis theory, which arose as a development of the Kovalevskaya-Painlevé analysis and Ziglin's integrability theory [72], [73]. By connecting the existence of first integrals with branching of solutions as functions of complex time to a property of the differential Galois group of a variational equation, it yields an effective method of proving non-integrability and detecting possible integrability of dynamical systems. Consider the system of holomorphic differential equations

$$\dot{x} = v(x), \qquad t \in \mathbb{C}, \quad x \in M, \tag{1}$$

defined on a complex n-dimensional manifold M. If $\phi(t)$ is a non-equilibrium solution of (1), then the maximal analytic continuation of $\phi(t)$ defines a Riemann surface Γ with t as a local coordinate. Together with (1) we consider its variational equations (VEs) restricted to $T_\Gamma M$, i.e.

$$\dot{\xi} = T(v)\xi, \qquad \xi \in T_\Gamma M.$$

We can decrease the order of that system by considering the induced system on the normal bundle $N := T_\Gamma M/T\Gamma$ of Γ:

$$\dot{\eta} = \pi_\star(T(v)\pi^{-1}\eta), \qquad \eta \in N$$

where $\pi : T_\Gamma M \to N$ is the projection. The system of $s = n - 1$ equations obtained in that way yields the so-called normal variational equations (NVEs). Their monodromy group $\mathcal{M} \subset \mathrm{GL}(s, \mathbb{C})$ is the image of the fundamental group $\pi_1(\Gamma, t_0)$ of Γ obtained in the process of continuation of the local solutions defined in a neighborhood of t_0 along closed paths with base point t_0. A non-constant rational function $f(z)$ of s variables $z = (z_1, ..., z_s)$ is called an integral (or invariant) of the monodromy group if $f(g \cdot z) = f(z)$ for all $g \in \mathcal{M}$. In his two fundamental papers [72], [73], Ziglin showed that if (1) possesses a meromorphic first integral, then $\mathcal{M}$ has a rational first integral. Ziglin found a necessary condition for the existence of a maximal number of first integrals (without involutivity property) for analytic Hamiltonian systems, when $n = 2m$, in the language of the monodromy group. Namely, let us assume that there exists a non-resonant element $g \in \mathcal{M}$. If the Hamiltonian system with m degrees of freedom has m meromorphic first integrals $F_1 = H, ..., F_m$, which are functionally independent in a connected neighborhood of Γ, then any other monodromy matrix $g' \in \mathcal{M}$ transforms eigenvectors of g to its eigenvectors.

There is a problem however in making that theory algorithmic: the monodromy group is known only for a few differential equations. To overcome that problem, Morales-Ruiz and Ramis recently generalized Ziglin's approach by replacing the monodromy group $\mathcal{M}$ by the differential Galois group $\mathcal{G}$ of the NVEs. They formulated [51] a new criterion of non-integrability for Hamiltonian systems in terms of the properties of the connected identity component of $\mathcal{G}$: if a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighborhood of the analytic curve Γ, then the identity component of the differential Galois group of NVEs associated with Γ is Abelian. Since $\mathcal{G}$ always contains $\mathcal{M}$, the Morales-Ramis non-integrability theorem always yields stronger necessary conditions than the Ziglin criterion.

When applying the Morales-Ramis criterion, our first step is to find a non-equilibrium particular solution, which often lies on an invariant submanifold. Next, we calculate the corresponding VEs and NVEs. If we know that our Hamiltonian system possesses k first integrals in involution, then we can consider VEs on one of their common levels, and the order of NVEs is equal to $s = 2(m - k)$ [22], [23]. In the last step we have to check if the identity component of $\mathcal{G}$ is Abelian, a task where the tools of algorithmic Galois theory (such has the Kovacic algorithm) become useful. In practice, we often check only whether that component is solvable (which is equivalent to check whether the NVEs have Liouvillian solutions), because the system is not integrable when that component is not solvable.

Our main objectives in that field are: (i) to apply the Morales-Ramis theory to various dynamical systems occurring in mechanics and astronomy; (ii) to develop algorithms that carry out effectively all the steps of that theory; (iii) to extend it by making use of non-homogeneous variational equations; (iv) to generalize it to various non-Hamiltonian systems, e.g. for systems with certain tensor invariants; (v) to formulate theorems about partial integrability of dynamical systems and about real integrability (for real dynamical systems) in the framework of the Morales-Ramis theory;

3.3. Mathematical web services

Keywords: *Computer algebra*, *MathML*, *OpenMath*, *Web services*, *communication*, *deductive databases*, *formula databases*.

The general theme of this aspect of our work is to develop tools that make it possible to share mathematical knowledge or algorithms between different software systems running at arbitrary locations on the web.

Most computer algebra systems deal with a lot of non algorithmic knowledge, represented directly in their source code. Typical examples are the values of particular integrals or sums. A very natural idea is to group this knowledge into a database. Unfortunately, common database systems are not capable to support the kind of mathematical manipulations that are needed for an efficient retrieval (doing pattern-matching, taking into account commutativity, etc.). The design and implementation of a suitable database raise some interesting problems at the frontier of computer algebra. We are currently developing a prototype for such a database that is capable of doing some deductions. Part of our prototype could be applied to the general problem of searching through mathematical texts, a problem that we plan to address in the near future.

The computer algebra community recognized more than ten years ago that in order to share knowledge such as the above database on the web, it was first necessary to develop a standard for communicating mathematical objects (via interprocess communication, e-mail, archiving in databases). We actively participated in the definition of such a standard, OpenMath (partly in the course of a European project). We were also involved in the definition of MathML by the World Wide Web Consortium. The availability of these two standards are the first step needed to develop rich mathematical services and new architectures for computer algebra and scientific computation in general enabling a transparent and dynamic access to mathematical components. We are now working towards this goal by experimenting with our mathematical software and emerging technologies (Web Services) and participating in the further development of OpenMath.

4. Application Domains

4.1. Panorama

We have applied our algorithms and programs for computing differential Galois groups to determine necessary conditions for integrability in mechanical modeling and astronomy. We also apply our research on partial linear differential equations to control theory, for example for parameterization and stabilization of linear control systems. Other applications include biology, where our algorithms for solving recurrence equations have been applied to the steady-state equations of nonhomogeneous Markov chains modeling the evolution of microsatellites in genomes.

5. Software

5.1. Maple package diffalg

Keywords: *analysis of singular solutions, differential algebra, differential elimination, nonlinear differential systems, triangulation-decomposition algorithms*.

Participant: Evelyne Hubert.

The *diffalg* library has been part of the commercial release of Maple since Maple V.5 (with an initial version of F. Boulier) and has evolved up to Maple 7. The library implements a triangulation-decomposition algorithm for polynomially nonlinear systems and tools for the analysis of singular solutions.

The recent development include an implementation of triangulation-decomposition algorithms for differential polynomial rings where derivations satisfy nontrivial commutation rules [10]. This new version also incorporates a specific treatment of parameters as well as improved algorithms for higher degree polynomials [42].

5.2. Library OreModules of Mgfun

Keywords: *Constructive algebraic analysis, Gröbner bases, Ore algebras, linear systems*.

Participants: Frédéric Chyzak, Alban Quadrat [correspondent], Daniel Robertz.

The library OREMODULES of MGFUN is dedicated to the study of over-/under-determined linear systems over some Ore algebras (e.g., ordinary differential equations, partial differential equations, differential time-delay equations, discrete equations) and their applications in mathematical physics (e.g., research of potentials, computations of the field equations and the conservation laws) and in linear control theory (e.g., controllability, observability, parameterizability, flatness of multidimensional systems with varying coefficients). The main novelty of OREMODULES is to use the recent development of the Gröbner bases over some Ore algebras (non-commutative polynomial rings) in order to effectively check some properties of module theory (e.g., torsion/torsion-free/reflexive/projective/stably free/free modules) and compute some important tools of homological algebra (e.g., free resolutions, split exact sequences, duality, extension functor, projective dimension, Krull dimension, Hilbert power series).

A library of examples (e.g., a two pendulum mounted on a car, time-varying systems, systems of algebraic equations, a wind tunnel model, a two reflector antenna, an electric transmission line, Einstein equations, Maxwell equations, linear elasticity, Lie-Poisson structures) has been developed in order to illustrate the main functions of OREMODULES.

The third release of OREMODULES will be soon available with new functions, a larger library of examples and the new package STAFFORD.

5.3. Library libaldor

Keywords: *Aldor*, *arithmetic*, *data structures*, *standard*.

Participant: Manuel Bronstein.

The LIBALDOR library, under development in the project for several years, became the standard ALDOR library, bundled with the compiler distribution since 2001.

5.4. Library Algebra

Keywords: *commutative algebra*, *computer algebra*, *linear algebra*, *polynomials*.

Participants: Manuel Bronstein, Marc Moreno-Maza [University of Western Ontario].

The ALGEBRA library, written in collaboration with Marc Moreno-Maza, is now also bundled with the ALDOR compiler, starting with version 1.0.2, which has been released in May 2004. It is intended to become a standard core for computer algebra applications written in ALDOR.

5.5. Library Σ^{it}

Keywords: *computer algebra*, *difference equations*, *differential equations*, *systems*.

Participant: Manuel Bronstein.

The Σ^{it} library contains our algorithms for solving functional equations. The stand-alone solver powers the the web service at http://www-sop.inria.fr/cafe/Manuel.Bronstein/sumit/bernina_demo.html.

6. New Results

6.1. Fast algorithms for linear differential and difference equations

Keywords: *Linear differential and difference equations*, *baby step-giant step*, *binary splitting*, *complexity*, *indefinite and definite summation*, *polynomial and rational solutions*.

Participants: Thomas Cluzeau, Alin Bostan [projet ALGO], Frédéric Chyzak [projet ALGO], Bruno Salvy [projet ALGO].

The search for rational (and thus polynomial) solutions of linear differential and difference equations lies at the heart of several important algorithms in computer algebra.
In [16], we investigate the problem of computing polynomial solutions of linear differential equations. Even for detecting the non-existence of non-zero polynomial solutions of a linear differential equation, there exists no general algorithm in polynomial complexity. However, high degree polynomial solutions are controlled by the differential equation. This allows to represent them with a compact data-structure: a linear recurrence and initial conditions. This compact representation enables us to compute polynomial solutions in quasi-optimal complexity (that is, optimal complexity modulo logarithm terms) using binary splitting and baby step/giant

step techniques. The timings of our implementations in Maple and Magma confirm the complexity estimates in practice.

We then addressed the problem of computing rational solutions. The compact representation of polynomial solutions of linear differential equations also allows to evaluate, in quasi-optimal complexity, their value and that of their derivatives at a rational or algebraic point. We can also perform divisions by high powers of a monomial which leads directly to an algorithm in quasi-optimal complexity to compute rational solutions of linear differential equations.

We now continue our work by looking at the case of difference equations. The calculation of polynomial solutions works in the same way except that the coefficients of these polynomials satisfy a linear recurrence equation in the binomial basis instead of the monomial basis. The generalization of our techniques for computing rational solutions is a little bit more difficult because of the appearance of a new phenomenon called the dispersion. However, binary splitting and baby step/giant step techniques together with the compact representation of the numerator allows us to obtain an algorithm in quasi-optimal complexity. As a direct consequence, we obtained fast versions of Abramov's algorithm computing rational solutions of linear recurrence equations, Gosper's algorithm for indefinite summation, and Zeilberger's algorithm for definite summation.

6.2. Algorithms for solving and factoring linear partial functional systems

Keywords: *Linear (partial) functional systems, Picard–Vessiot extensions, decomposition, factorization, modules over Laurent-Ore algebras, polynomial solutions, rational solutions.*

Participants: Manuel Bronstein, Ziming Li, Min Wu.

Linear (partial) functional systems consist of linear partial differential or (q)-difference equations and any mixture thereof. We continued our work towards developing effective algorithms and programs to solve and factor linear functional systems with finite-dimensional solution spaces (called fLFS).

Our research involves two basic issues in the study of fLFS: constructing a suitable ring that contains all solutions of a system, and factoring a system into "subsystems" with lower-dimensional solution spaces.

In order to check in practice whether a linear functional system has finite-dimensional solution space and to compute its linear dimension, we present two algorithms [7]: one is based on Gröbner basis techniques in Laurent-Ore algebras and is applicable to general systems; the other is based merely on the Gröbner basis computation in Ore algebras and linear algebra, and is applicable to the system with finite rank in the Ore setting.

By introducing the notion of modules of formal solutions, we generalize the notion of Picard-Vessiot extensions from linear ordinary differential/difference equations [66], [65] to fLFS and prove the existence of Picard-Vessiot extensions [17]. We also give an algorithmic approach to completing partial solutions of fLFS, which points out how to solve an fLFS when its factors are known.

We interpret the factorization problem of fLFS using their modules of formal solutions. This is a generalization of the cases of linear ODEs [65] and of D-finite systems of linear PDEs [15]. Based on this interpretation, we present an algorithm [7] for finding all submodules of a finite-dimensional module over a Laurent-Ore algebra. This algorithm unifies several known algorithms for factoring D-finite systems of linear PDEs and for factoring linear ordinary difference equations in a module-theoretic setting.

6.3. Formal solutions of partial differential systems at a singular point

Keywords: *Pfaffian systems, normal crossings singularities, regular and irregular singularity.*

Participants: Nicolas Le Roux, Moulay Barkatou [Université de Limoges], Evelyne Hubert.

Completely integrable systems are well studied. They are encountered for instance in isomonodromic deformations or in differential geometry. We consider here completely integrable systems with normal crossings singularities. These systems can be rewritten locally in the form

$$(\star)\left\{\begin{array}{l}\frac{\partial Y}{\partial x_1}=x_1^{-(p_1+1)}A_1Y\\ \vdots\\ \frac{\partial Y}{\partial x_n}=x_n^{-(p_n+1)}A_1Y\end{array}\right.$$

The A_i are $m\times m$ matrices with entries in the ring $\mathbb{C}[[x_1,...,x_n]]$ and the p_i are non negative integers. Y is a $m\times 1$ vector of unknowns with coordinates in an extension of $\mathbb{C}((x_1,...,x_n))$.

We investigate algorithms for the computation of formal solutions of completely integrable systems with normal crossing singularities. We expect to extend the known algorithms for ordinary differential systems.

As in the ordinary case, there are three kinds of singularities at the origin. Singularities of the *first kind* are those for which $p_i = 0$ for all i. Every system of the first kind is equivalent, by a transformation $Y = TZ$ with $T \in GL_m(\mathbb{C}((x)))$, to a system where the matrices have constant entries [39], [63]. These constant matrices are the monodromy of the system. Singularities are *regular* if the system is equivalent to one of the first kind. All other singularities are called *irregular*. There are algorithms to decide whether a singularity is regular that is based on a criterion by Deligne [30] and Van Den Essen [32]. Yet this is not constructive as the algorithm does not produce the transformation.

For singularities of the first kind the solution space can be described in terms of the monodromy matrices. The description of the solution space at regular singularities is then obtained if we have the transformation that reduces the system to a system of the first kind. For irregular singularities, Charrière made the first attempt at describing the solution space in the case where the system only involves two independent variables [26]. She showed that Turritin's form for linear differential system [71] can be carried to this case. Her proof uses the lattice approach initiated by Gérard & Levelt [38], [39]. Next the result was extended to the general case independently by Levelt & Van Den Essen [33] and Charrière & Gérard [27].

Our first goal is to provide an algorithm for reducing the order of the singularity at the origin - that is the length of $(p_1,...,p_n)$ - by linear transformations. Such a reduction process would allow us to decide whether or not the singularity is regular. We have investigated the possibility to extend Moser's algorithm [52], [24] and Levelt's algorithm [24] that work in the ordinary case. We clarified the relationship between those two algorithms. The presentation with lattices by Levelt seems easier to extend. We have shown that it can be extended to the case $n = 2$. This work is to be submitted to the international conference ISSAC 2006. It is also implemented in Maple. We have used parts of the implementation of Moser's algorithm by M. Barkatou [24].

For $n > 2$, the difficulties we encounter are twofold. First, the systems considered are not invariant under transformation $Y = TZ$. Secondly, there is no warranty that the lattices are free. The same problem arose in [33], [27].

6.4. Linear systems over Ore algebras and applications

Keywords: *(over-/under-determined) linear systems, Algebraic analysis, Gröbner bases for non-commutative polynomial rings, Ore algebras, multidimensional systems.*

Participants: Alban Quadrat, Daniel Robertz.

In [8], we study the structural properties of linear systems over Ore algebras (e.g., ordinary differential equations, differential time-delay equations, partial differential equations, difference equations). Using the recent development of Gröbner bases over some Ore algebras (i.e., non-commutative polynomial rings), we show how to make some important concepts of homological algebra (e.g., free resolutions, split exact sequences, duality, extension functor, projective dimension, Krull dimension, Hilbert power series) constructive. Using these results, we obtain constructive algorithms which check the structural properties of under-determined linear systems over Ore algebras described within a module-theoretic language (e.g. torsion/torsion-free/reflexive/projective/stably free/free modules). In particular, we explain how these properties are related to the existence of successive parameterizations of all the solutions of a system. Moreover, we show how these properties generalize the well-known concepts of primeness developed in the literature of multidimensional systems for systems with constant coefficients. Then, using a dictionary between the structural properties of under-determined linear systems over Ore algebras and some concepts of control theory, we show that the previous algorithms allow us to effectively check whether or not a linear control system over certain Ore algebras is (weakly, strongly) controllable, observable, parameterizable, flat, π-free... We note that the problem of parameterizing all the solutions of controllable multidimensional linear systems has been extensively studied by the school of M. Fliess in France [35], [36] and J. C. Willems [59] in the Netherlands. Hence, our results allow us to constructively answer that problem for a large class of multidimensional systems considered in the literature.

The different algorithms obtained in [8] have been implemented by D. Robertz and A. Quadrat in the second release of the library OREMODULES (5.2). The main advantage of the language of homological algebra used in [8] carries over the implementations in OREMODULES: up to the choice of the domain of functional operators which occurs in a given system, all algorithms are stated and implemented in sufficient generality such that linear systems defined over the Ore algebras developed in the Maple package of *Ore_algebra* are covered at the same time. This approach is explained in [9] and illustrated on different systems (e.g., a linearized pendulum, a time-varying differential algebraic system, a two reflector antenna and a vibrating string with an interior mass).

The purpose of [20] is to study the so-called *Monge problem* for linear systems over some classes of Ore algebras. This problem was first studied by J. Hadamard and E. Goursat for systems of partial differential equations in the period 1900-1930. Using the algebraic analysis approach developed in [8], we derive a necessary and sufficient condition so that we can find a *Monge parametrization* of all the solutions of an underdetermined linear system over an Ore algebra obtained by gluing the autonomous elements of the system to the parameterizable subsystem. The autonomous elements and the parameterizable subsystem can be computed by means of the algorithms developed in [8]. We point out that a Monge parametrization is more general than the parameterizations used in [8] as it depends on arbitrary functions of a certain number of the independent variables whereas we only consider parameterizations depending on all the independent variables in [8]. Effective algorithms checking the given condition are obtained and implemented in OREMODULES. These results are in particular useful if we want to parametrize the solutions of the Saint Venant equations or some linear systems of partial differential equations appearing in the theory of elasticity.

It is well-know that an analytic time-varying controllable ordinary differential system is *flat* outside some singularities [50]. In [19], we prove that every analytic time-varying controllable linear system is a projection of a flat system. We give an explicit description of the flat system which projects onto a given controllable one. This phenomenon is similar to a classical one largely studied in algebraic geometry and called the *blowing-up* of a singularity. These results simplify the ones obtained in [50] and generalize them to multidimensional linear systems over an Ore algebra. Finally, we prove that every controllable multi-input ordinary differential linear system with polynomial coefficients is flat, answering a question asked by K. B. Datta.

A well-known and difficult result due to J. T. Stafford asserts that a stably free left module M over the Weyl algebras $D = A_n(k)$ or $B_n(k)$ of differential operators with polynomial/rational coefficients $-$ where k is a field of characteristic 0 $-$ with $\mathrm{rank}_D(M) \geq 2$ is free. The purpose of [14] is to present a constructive proof of this result as well as an effective algorithm for the computation of bases of M. This algorithm, based on the new constructive proofs [41], [48] of J. T. Stafford's result on the number of generators of left ideals over D, performs Gaussian eliminations on the columns of the formal adjoint of the presentation matrix of M. Moreover, we show that J. T. Stafford's result is a particular case of a more general one asserting that a stably free left D-module M with $\mathrm{rank}_D(M) \geq \mathrm{sr}(R)$ is free, where $\mathrm{sr}(R)$ denotes the stable range of a ring D. This result is constructive if the stability of unimodular vectors with entries in D can be tested. Finally, an algorithm which computes the left projective dimension of a general left D-module M defined by means of a finite free resolution is presented. In particular, it allows us to check constructively whether or not the left D-module M is stably free. All these algorithms have recently been implemented in the package STAFFORD based on OREMODULES. Hence, we can now effectively compute the two generators of any left ideal over D and a basis of any free left D-module. These results can be used in order to recognize whether or not a linear system of ordinary/partial differential equations with polynomial/rational coefficients admits an injective parametrization. This last result is in particular interesting in the theory of flat systems [35], [36], [50].

6.5. Algebraic analysis approach to infinite-dimensional linear systems

Keywords: *Systems of partial differential equations or differential time-delay equations, analysis and synthesis problems, internal/strong/simultaneous/robust/optimal stabilization, module theory*.

Participant: Alban Quadrat.

For single-input single output (SISO) linear systems, we show in [12] that a module-theoretic duality exists between the operator-theoretic approach to analysis and synthesis problems developed in the literature of infinite-dimensional systems (see [29] and the references therein) and the fractional ideal approach recently developed in [60]. In particular, this new duality helps us to understand how the algebraic properties of systems are reflected by the operator-theoretic approach and conversely. In terms of modules, we characterize the domain and the graph of an internally stabilizable plant or that of a plant which admits a (weakly) coprime factorization. Moreover, we prove that internal stabilizability implies that the graph of the plant and the one of a stabilizing controller are the two direct summands of the global signal space. These results generalize those obtained in the literature. Finally, we exhibit a class of signal spaces over which internal stabilizability is equivalent to the existence of a bounded inverse for the linear operator mapping the errors e_1 and e_2 of the closed-loop system to the inputs u_1 and u_2.

The purpose of the paper [11] is to develop a lattice approach to analysis and synthesis problems. This new mathematical framework allows us to give simple and tractable necessary and sufficient conditions for internal stabilizability and for the existence of (weakly) left-/right-/doubly coprime factorizations of multi-input multi-output (MIMO) (infinite-dimensional/multidimensional) linear systems. These results extend the ones recently obtained in [60] for SISO systems. In particular, we prove that every internally stabilizable multidimensional system in the sense of the structural stability admits doubly coprime factorizations, solving the Lin's well-known conjecture in the literature of multidimensional systems.

Within the lattice approach to analysis and synthesis problems developed in [11], we obtain in [13] a general parametrization of all stabilizing controllers for internally stabilizable MIMO plants which do not necessarily admit doubly coprime factorizations. This parametrization is a linear fractional transformation of free parameters and the set of arbitrary parameters is characterized. This parametrization generalizes for MIMO plants the parametrization recently obtained in [60] for SISO plants. In particular, if the plant admits a doubly coprime factorization, we then prove that this general parametrization becomes the classical *Youla-Kučera parameterization* of all stabilizing controllers. Finally, we show that this parameterization also allows us to rewrite the standard problem of finding the robust stabilizing controllers as affine, and thus, convex problems. A self-contained proof of the existence of this general parametrization all stabilizing controllers is also presented in the congress paper [18].

6.6. Jacobson and Hermit forms: Implementations and applications

Participants: Grégory Culianez [INSA de Toulouse], Alban Quadrat.

The purpose of this internship was to implement the Hermite and Jacobson forms over some classes of non-commutative euclidean domains (e.g., $\mathbb{Q}(t)\left[\frac{d}{dt}\right]$, $\mathbb{Q}(n)[\sigma]$, $\mathbb{Q}(x,y)[\partial_x]$) in the library OREMODULES. Indeed, the Hermite and Smith forms only exist in Maple for some classes of commutative euclidean domains. This implementation will be added to the next release of OREMODULES. Finally, applications of these canonical forms for the study of general time-varying linear systems of ordinary differential equations and shift-variant linear systems of univariate discrete equations, as well as, in control theory (e.g., controllability, observability, parameterizable) are demonstrated in G. Culianez's final report available at www-sop.inria.fr/cafe/Alban.Quadrat/stages.html.

6.7. Implementation of the Quillen-Suslin theorem in OreModules

Participants: Job Evers [MIT], Alban Quadrat.

In 1955, a famous conjecture due to Serre asserted that if we take a row vector v with coefficients in a commutative polynomial ring $A = k[x_1, ..., x_n]$ (k is a field) which admits a right-inverse over A, then v can be embedded into a square unimodular matrix, namely, a matrix with a non-zero constant determinant. This conjecture was proved independently by Quillen and Suslin in 1976 and is nowadays called the Quillen-Suslin theorem. In terms of module theory, Serre's conjecture means that a projective A-module is free and the computation of the unimodular matrix then gives an explicit basis for this module. Constructive proofs of the Quillen-Suslin theorem have been largely studied in the literature of constructive algebra. The purpose of this internship was to start implementing in OREMODULES the difficult computations of bases for projective A-modules. This project was mainly achieved in time and the missing part of the algorithm has recently been completed by A. Fiabanska (University of Aachen, Germany) in the framework of her PhD thesis partially supervised by A. Quadrat. A package called QUILLEN-SUSLIN will be soon added to OREMODULES. The Quillen-Suslin theorem has many applications in mathematical systems theory and, in particular, it shows a way for the computation of flat outputs of flat shift-invariant multidimensional linear systems such as time-invariant differential time-delay control systems. Hence, the present works gives the first implementation for the computation of flat outputs of flat shift-invariant multidimensional linear systems. Applications of such a class of systems were largely demonstrated in the literature of control theory and, in particular, for the motion planning and tracking problems (e.g., see the library of examples of OREMODULES). Finally, these constructive results have recently allowed A. Fabianska and A. Quadrat to show that every flat shift-invariant multidimensional linear system is equivalent to a controllable 1-D linear system obtained by setting to 0 all but one functional operators in the system matrix. Moreover, it was proved that a flat differential time-delay linear system is equivalent to the controllable ordinary differential linear system without delays, i.e., the system obtained by setting the time-delay amplitude to 0. The implementation of the Quillen-Suslin theorem gives a constructive way to get the precise invertible transforms between these systems. Applications of these results in the study of stabilization problems will be shown in forthcoming publications.

6.8. Algebraic construction and structure of differential invariants

Participants: Evelyne Hubert, Irina Kogan [North Carolina State University].

The theory of differential invariants [56] and algebraic invariants [70] seem to have developed independently. Motivated by a wealth of applications [31], [37], [55], [57] both have become in recent years the subject of computational mathematics [31], [34], [68] yet remained separate topics. Differential invariants are intimately linked with physics, and more generally the study of differential systems, while an algebraic theory provides the proper ground to symbolic algorithms.

Inspired by the geometric construction of [34], we provided a new and simple algorithm to compute a generating set of rational invariants of a rational action [21]. The generating invariants come together with an algorithm to rewrite any rational invariant in terms of them. They appear as the coefficients of the Gröbner basis of the ideal of the graph of the action or only a section of it. This simple algorithm, based on algebraic elimination, is easily implemented and can take advantage of the efficient symbolic software developed for algebraic geometry. It opens up computational projects in order to address long standing challenges, for instance in classical invariant theory.

The second variant of the above construction relies on the choice of a *cross-section* to the orbits, i.e. an irreducible algebraic variety that is generically transverse to the orbits. A generic linear affine space of the appropriate dimension can take up this role. We show that the field of rational functions on the cross-section to the orbits is an algebraic extension of the field of rational invariants. We describe this extension by a prime zero dimensional ideal over the field of rational invariants of which we can compute a set of generators. This is thus a constructive version of results in [70]. On the other hand, we establish the bridge between this algebraic vision and the moving frame method of [34] in differential geometry, enhancing thereby this latter with a fully algorithmic construction circumventing the use of the implicit function theorem.

The *replacement invariants* we introduce as the zeros of the prime zero dimensional ideal are the algebraic analogues of Cartan's *normalized invariants*. Rewriting any rational invariant in terms of a replacement invariant is a trivial substitution. We thus have a new class of algebraic invariants having the structure and properties that could be exploited from a computational point of view. Seeing invariants as functions on the cross-section also provides the basis for our investigation of the structure of differential invariants as a differential ring. Other questions in algebraic geometry and of practical interest are raised by this construction.

7. Other Grants and Activities

7.1. National initiatives

A. Quadrat is a member of the working group “Systèmes à Retards” of the *GdR Automatique*.

7.2. European initiatives

7.2.1. ECO-NET Program

The ECO-NET program, grant number 08119TG, was extended to 2005. This is for a collaboration between Prof. S.A. Abramov (Moscow), Prof. M. Petkovšek (Ljubljana), and M. Bronstein (CAFE). His program is now continued with A. Quadrat.

7.2.2. PAI Amadeus

A collaboration with RISC (Research Institute in Symbolic Computation) in Hagenberg, Austria was funded by the PAI program AMADEUS. The coordinators are Ralf Hemmecke for the Austrian side and Evelyne Hubert replaced Manuel Bronstein in June for the French side. One objective of the collaboration is to develop and implement a very generic and efficient engine to compute Gröbner basis with ideals and modules over rings of linear operators, such as Ore algebras or Poincare-Birkhoff-de Witt algebras

7.3. Other international initiatives

7.3.1. Actions of the European and International Affairs Department

The project A-Lie-Gorithm to collaborate with P. Olver, University of Minneapolis, on algebraic algorithms for determining the structure of Lie pseudo-groups, was accepted for funding after the *Mini-call "USA, Scandinavian countries, Taiwan"*http://www-direction.inria.fr/international/PROGRAMMES/miniappel.eng.html.

7.3.2. China

Following the termination of our PRA with Z. Li (Academia Sinica), our collaboration continues with the co-direction of the thesis of Min Wu, who is alternating 6-month stays in our project and in Beijing.

7.4. International networks and working groups

Stéphane Dalmas is a member of the Math Interest Group of the World Wide Web Consortium. This group is responsible for maintaining existing documents related to MathML, working with other W3C groups and provide general support on MathML and mathematics on the Web. Stéphane Dalmas was a member of the W3C Math Working Group, the group responsible for defining MathML, an XML application for describing mathematical notation and capturing both its structure and content. The goal of MathML is to enable mathematics to be served, received, and processed on the World Wide Web, just as HTML has enabled this functionality for text.

7.5. Visiting scientists

7.5.1. France

Jacques-Arthur Weil, University of Limoges, visited February 1-4th, and September 11-15th 2005 for collaboration with team members.

Marc Rybowicz , University of Limoges, visited from May 30th to June 10th, 2005 to collaborate with M. Bronstein on integration.

J.-F. Pommaret, Ecole Nationale des Ponts et Chaussées (CERMICS), has been invited for a week for a series of lectures entitled “An introduction to the Galois theory for partial differential equations” (October 3-7th, 2005) and for a collaboration with A. Quadrat.

M. Barakatou and N. Le Roux, University of Limoges, visited October 24-28th, 2005 for collaboration with E. Hubert.

7.5.2. Europe

D. Robertz, University of Aachen, visited A. Quadrat from March 1st to April 8th. They have collaborated on their OREMODULES project in the framework of D. Robertz’s PhD thesis.

Ralf Hemmecke (RISC Linz, Austria) visited our project for one week in October in order to collaborate on the design of the LIBALDOR and ALGEBRA libraries (see 5.3) to include non-commutative polynomials. The visit is funded by the PAI Amadeus (section 7.2).

7.5.3. Outside Europe

Rouchdi Bahloul, Kobe University, visited January 4-5th, 2005.

Peter Olver, University of Minneapolis, visited June 20-29, 2005, for collaboration with E. Hubert on the A-Lie-Gorithm (see 7.3).

8. Dissemination

8.1. Leadership within scientific community

- M. Bronstein was a member of the editorial boards for the *Journal of Symbolic Computation* and for the *Algorithms and Computation in Mathematics* Springer monograph series.
- M. Bronstein was vice-chair of the SIGSAM special interest group of ACM.
- Evelyne Hubert is member of the council of the association *femmes & mathématiques*. As such she participates to the monthly meetings of the council to propose, discuss and undertake the actions of the association. Those aim at promoting scientific education and careers for women.
- Evelyne Hubert is the founding chair of the *Séminaires Croisés*. The *séminaires croisés* is a unique initiative of global scientific animation within the site of INRIA Sophia Antipolis. It is now organized by the service REV.
- Evelyne Hubert is a member of the COLORS committee chaired by Rose Dieng.

8.2. Teaching

- Evelyne Hubert taught a course of practical computer algebra in Master in Numerical Mechanics (March 2005) and in ISIA (April 2005) both being affiliated to the *Ecole des Mines de Paris*.
- Evelyne Hubert taught computer science and computer algebra in the *classes préparatoires scientifiques* at the *Centre International de Valbonne*.
- Evelyne Hubert was invited to present a tutorial course at the International Symposium of Symbolic and Algebraic Computation (Beijing, China, July 2005).

8.3. Dissertations and internships

Doctorates completed in 2004:

1. Min Wu, University of Nice and Chinese Academy of Sciences: *On Solutions of Linear Functional Systems and Factorization of Modules over Laurent-Ore Algebras*. Co-directed by Manuel Bronstein (CAFE) and Ziming Li (Chinese Academy of Sciences).

Doctorates in progress in the project:

1. Nicolas Le Roux, University of Limoges: *Local study of systems of partial differential equations*. Co-directed by Moulay Barkatou (University of Limoges) and Evelyne Hubert.
2. Daniel Robertz is partially supervised by A. Quadrat in the framework with his PhD thesis at the University of Aachen (Germany) with Prof. W Plesken.
3. Anna Fabianska is partially supervised by A. Quadrat in the framework with her PhD thesis at the University of Aachen (Germany) with Prof. W Plesken.

Internships completed in 2005:

1. Job Evers, Internship from MIT: *Implementation of the Quillen-Suslin theorem in* OREMODULES, supervised by Alban Quadrat (June-August 2005)
2. Grégory Culianez, Internship from ENSTA: *Hermit and Jacobson forms: Implementation and applications*, supervised by Alban Quadrat (June-July 2005)

8.4. Conferences and workshops, invited conferences

M. Bronstein was invited to give a colloquium talk at RISC, Austria.

E. Hubert participated and gave submitted talks at the MEGA conference in Sardinia and the Journees National de Calcul Formel in Luminy. She presented a poster at the ISSAC conference which received the best poster award. She was invited to participate and give talks in the Symbolic Analysis workshop of the FoCM conference in Spain and the workshop Challenges in Linear and Polynomial Algebra in Symbolic Computation Software held in Alberta, Canada.

E. Hubert was invited to visit University of Kent (UK, March 20-27th, 2005) by E. Mansfield. She visited P. Olver at the University of Minneapolis (Minnesota) in the frame of the collaboration A-Lie-Gorithm (see 7.3). At this occasion she was invited to present her work on algebraic invariants at the Algebraic Geometry seminar at the School of Mathematics.

A. Quadrat presented his recent work at the 16th IFAC World Congress (2 papers) in Prague (Czech Republic) and a join paper will be presented by D. Robertz at the 44th IEEE Conference on Decision and Control and European Control Conference CDC-ECC 2005 in Seville (Spain). Moreover, A. Quadrat will give a talk at the Journées Nationales de Calcul Formel in Luminy (France) as well as one talk at the Research Institute for Symbolic Computation (RISC-Linz, Austria) while visiting R. Hemmecke in December (PAI Amadeus).

A. Quadrat was invited to expose his work at the workshop Mathematics, Algorithms, Proofs (Dagstuhl, Germany), at the Séminaire de Calcul Formel de l'IRMAR (Rennes, France), at the NETCA Workshop Verification and Theorem Proving for Continuous Systems (Oxford, United Kingdom) and at the colloquium Graduiertenkolleg Hierarchie und Symmetrie in mathematischen Modellen of the University of Aachen (Germany). Finally, he will be invited to give two talks at the Kolloquium Mathematik of the University of Innsbruck (Austria) in December.

A. Quadrat was finally invited to present the contributions of C. Méray in the theory of systems of partial differential equations at the Colloque en hommage à Charles Méray, Mathématicien et Vigneron Chalonnais (1835-1911) in Chalon Sur Saône (France).

M. Wu gave a talk at University of Limoges (January 2005). She presented her work at RISC, Austria (Februrary 2005). She took part in the Ecole Jeunes Chercheurs en Algorithmique et Calcul Formel 2005 in Montpellier (April 4-8th). She gave a contributed talk on the joint work with Manuel Bronstein and Ziming Li at the ISSAC'2005 conference. She defended her PhD thesis in Beijing on July 28th, 2005.

T. Cluzeau was invited by the ALGORITHMS project of the INRIA to collaborate with A. Bostan, F. Chyzak and B. Salvy in october and november 2005. He gave a talk at the Journées Nationales de Calcul Formel in Luminy (France). T. Cluzeau joined the ECO-NET meeting (section 7.2) in Ljubljana (Slovenia) in december 2005. The object of this meeting is to work with S. Abramov and M. Petkovšek on the subject: computer algebra and q-hypergeometric terms.

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