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Project-Team Café

Calcul Formel et Équations

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1. Team

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2. Overall Objectives

2.1. Overall Objectives

We develop computer algebra methods and softwares for solving functional equations, i.e., equations where the unknowns represent functions rather than numerical values. We study in particular linear and nonlinear differential and (q)-difference equations, partial and ordinary. Solving is meant in the broad sense of extracting qualitative features of the solutions. Our goal is to foster the use of those symbolic methods in science and engineering by producing the programs and tools necessary to apply them to academic and industrial problems. Because symbolic computation is based on algebraic structures while our problems are analytic in nature, we develop the necessary theory for their algebraization.

3. Scientific Foundations

3.1. Differential ideals and D-modules

Keywords: *D-modules*, *algebraic analysis*, *control theory*, *differential algebra*, *differential elimination*, *differential systems*, *formal integrability*, *formal integrability*, *holonomic systems*, *involution*.

Algorithms based on algebraic theories are developed to investigate the structure of the solution set of general differential systems. Different algebraic and geometric theories are the sources of our algorithms.

3.1.1. Differential elimination and completion

Formal integrability is the first problem that our algorithms address. The idea is to *complete* a system of partial differential equation so as to be in a position to compute the *Hilbert differential dimension polynomial* or equivalently, its coefficients, the *Cartan characters*. Those provide an accurate measure of the arbitrariness that comes in the solution set (how many arbitrary functions of so many variables). Closely related is the problem of determining the *initial conditions* that can be freely chosen for having a well-posed problem (i.e. that lead to the existence and uniqueness of solutions). This is possible if we can compute all the differential relations up to a given order, meaning that we cannot obtain equations of lower order by combining the existing equations in the system. Such a system is called *formally integrable* and numerous algorithms for making systems of partial differential equations formally integrable have been developed using different approaches by E. Cartan [25], C. Riquier [77], M. Janet [49] and D. Spencer [84].

Differential elimination is the second problem that our algorithms deal with. One typically wants to determine what are the lowest differential equations that vanish on the solution set of a given differential system. The sense in which *lowest* has to be understood is to be specified. It can first be order-wise, as it is of use in the formal integrability problem. But one can also wish to find differential equations in a subset of the variables, allowing the model to be reduced.

The radical differential ideal generated by a set of differential polynomials $\mathcal{S}$, i.e. the left-hand side of differential equations where the right-hand side is zero, is the largest set of differential polynomials that vanish on the solution set of $\mathcal{S}$. This is the object that our algorithms manipulate and for which we compute adequate representations in order to answer the above questions.

In the nonlinear case the best we can hope for is to have information outside of some hypersurface. Actually, the radical differential ideal can be decomposed into components on which the answers to formal integrability and eliminations are different. For each component the *characteristic set* delivers the information about the singular hypersurface together with the quasi-generating set and membership test.

Triangulation-decomposition algorithms perform the task of computing a characteristic set for all the components of the radical differential ideal generated by a finite set of differential polynomials. References for those algorithms are the book chapters written by E. Hubert [4], [5]. They are based on the differential algebra developed by Ritt [78] and Kolchin [51].

The objectives for future research in the branch of triangulation-decomposition is the improvement of the algorithms, the development of alternative approaches to certain classes of differential systems and the study of the intrinsic complexity of differential systems.

Another problem, specific to the nonlinear case, is the understanding and algorithmic classification of the different behaviours of interference of the locus of one component on the locus of another. The problem becomes clear in the specific case of radical differential ideals generated by a single differential polynomial. One then wishes to understand the behaviour of non singular solutions in the vicinity of singular solutions. Only the case of the first order differential polynomial equations is clear. Singular solutions are either the envelope or the limit case of the non singular solutions and the classification is algorithmic [78], [46].

3.1.2. Algebraic analysis

Many systems coming from mathematical physics, applied mathematics and engineering sciences can be described by means of systems of ordinary or partial differential equations, difference equations, differential time-delay equations... In the case of a linear system $\mathcal{S}$, the system can be defined by means of matrices with entries in non-commutative algebras of functional operators such as differential operators, shift operators, time-delay operators... An important class of operators algebras is called *Ore algebras* and was introduced in [28].

The methods of *algebraic analysis* (algebraic D-modules) [22], [50], [66] give a way to intrinsically study a linear functional system by means of its associated finitely presented left module over an Ore algebra. Thanks to the works of B. Malgrange, V. Palamodov, J. Bernstein and M. Kashiwara, algebraic analysis yields new results and information about the algebraic and analytic properties of linear functional systems, their solutions and associated geometric invariants (e.g., characteristic varieties, dimensions).

Algebraic analysis is becoming algorithmic thanks to the recent development of Gröbner bases over Ore algebras (e.g., the Maple package *Ore_algebra* developed by F. Chyzak [28], SINGULAR: PLURAL developed by G.-M. Greuel, V. Levandovskyy and H. Schönemann [53]) and involutive bases in Ore algebras (e.g., the Maple package JANETORE developed by D. Robertz [79]), enabling the implementation of efficient algorithms for making systems formally integrable, computing special closed-form solutions [64], [65] or studying properties of linear functional systems by means of the algebraic properties of the underlying modules.

Within constructive algebraic analysis, it is now possible to algebraically study systems of linear functional equations, and our objectives in that field are: (i) to develop and implement efficient algorithms for computing the polynomial and rational solutions of such systems and, further, for factoring and decomposing their associated D-modules; (ii) to study the links between the algebraic and analytic properties of such systems (since the algorithmic determination of the algebraic properties yields information about the analytic properties); (iii) to apply the above algorithms to study problems coming from mathematical physics and control theory.

3.2. Groups of transformations

Keywords: *Hamiltonian mechanics*, *closed-form solutions*, *differential Galois group*, *differential invariants*, *dynamical systems*, *formal integrability*, *linear systems of partial differential equations*, *nonlinear differential systems*, *symmetry*, *variational equations*.

3.2.1. Lie groups of transformations

Though not a major subject of expertise, the topic is at the crossroads of the algorithmic themes developed in the team.

The Lie group, or symmetry group, of a differential system is the (biggest) group of point transformations leaving the solution set invariant.

Besides the group structure, a Lie group has the structure of a differentiable manifold. This double structure allows to concentrate on studying the tangent space at the origin, the Lie algebra.

The Lie group and the Lie algebra thus capture the geometry of a differential system. This geometric knowledge is exploited to *solve* nonlinear differential systems.

The Lie algebra is described by the solution set of a system of linear partial differential equations, whose determination is algorithmic. The dimension of the solution space of that linear differential system is the dimension of the Lie group and can be determined by the tools described in Section 3.1. Explicit subalgebras can be determined thanks to the methods developed within the context of Section 3.2.2.

For a given group of transformations on a set of independent and dependent variables there exist invariant derivations and a finite set of differential invariants that generate all the differential invariants [86]. This forms an intrinsic frame for expressing any differential system invariant under this group action.

This line of ideas took a pragmatic shape for computation with the general method of M. Fels and P. Olver [39] for computing the generating set of invariants. The differential algebra they consider has features that go beyond classical differential algebra. We are engaged in investigating the algebraic and algorithmic aspects of the subject.

3.2.2. Galois groups of linear functional equations

Differential Galois theory, developed first by Picard and Vessiot, then algebraically by Kolchin, associates a linear algebraic group to a linear ordinary differential equation or system. Many properties of its solutions, in particular the existence of closed-form solutions, are then equivalent to group-theoretic properties of the associated Galois group [82]. By developing algorithms that, given a differential equation, test such properties, Kovacic [52] and Singer [80] have made the existence of closed-form solutions decidable in the case of equations with polynomial coefficients. Furthermore, a generalization of differential Galois theory to linear ordinary difference equations [83] has yielded an algorithm for computing their closed-form solutions [44]. Those algorithms are however difficult to apply in practice (except for equations of order two) so that many algorithmic improvements have been published in the past 20 years. Our objectives in this field are to improve the efficiency of the basic algorithms and to produce complete implementations, as well as to generalize them and their building blocks to linear partial differential and difference equations.

An exciting application of differential Galois theory to dynamical systems is the Morales-Ramis theory, which arose as a development of the Kovalevskaya-Painlevé analysis and Ziglin's integrability theory [90], [91]. By connecting the existence of first integrals with branching of solutions as functions of complex time to a property of the differential Galois group of a variational equation, it yields an effective method of proving non-integrability and detecting possible integrability of dynamical systems. Consider the system of holomorphic differential equations

$$\dot{x} = v(x), \qquad t \in \mathbb{C}, \quad x \in M, \tag{1}$$

defined on a complex n-dimensional manifold M. If $\phi(t)$ is a non-equilibrium solution of (1), then the maximal analytic continuation of $\phi(t)$ defines a Riemann surface Γ with t as a local coordinate. Together with (1) we consider its variational equations (VEs) restricted to $T_\Gamma M$, i.e.

$$\dot{\xi} = T(v)\xi, \qquad \xi \in T_\Gamma M.$$

We can decrease the order of that system by considering the induced system on the normal bundle $N := T_\Gamma M / T\Gamma$ of Γ:

$$\dot{\eta} = \pi_{\star}(T(v)\pi^{-1}\eta), \qquad \eta \in N$$

where $\pi : T_\Gamma M \to N$ is the projection. The system of $s = n - 1$ equations obtained in this way yields the so-called normal variational equations (NVEs). Their monodromy group $\mathcal{M} \subset \mathrm{GL}(s, \mathbb{C})$ is the image of the fundamental group $\pi_1(\Gamma, t_0)$ of Γ obtained in the process of continuation of the local solutions defined in a neighbourhood of t_0 along closed paths with base point t_0. A non-constant rational function $f(z)$ of s variables $z = (z_1, ..., z_s)$ is called an integral (or invariant) of the monodromy group if $f(g \cdot z) = f(z)$ for all $g \in \mathcal{M}$. In his two fundamental papers [90], [91], Ziglin showed that if (1) possesses a meromorphic first integral, then $\mathcal{M}$ has a rational first integral. Ziglin found a necessary condition for the existence of a maximal number of first integrals (without involutivity property) for analytic Hamiltonian systems, when $n = 2m$, in the language of the monodromy group. Namely, let us assume that there exists a non-resonant element $g \in \mathcal{M}$. If the Hamiltonian system with m degrees of freedom has m meromorphic first integrals $F_1 = H, ..., F_m$, which are functionally independent in a connected neighbourhood of Γ, then any other monodromy matrix $g' \in \mathcal{M}$ transforms eigenvectors of g into its eigenvectors.

There is a problem however in making that theory algorithmic: the monodromy group is known only for a few differential equations. To overcome that problem, Morales-Ruiz and Ramis recently generalized Ziglin's approach by replacing the monodromy group $\mathcal{M}$ by the differential Galois group $\mathcal{G}$ of the NVEs. They formulated [61] a new criterion of non-integrability for Hamiltonian systems in terms of the properties of the connected identity component of $\mathcal{G}$: if a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic curve Γ, then the identity component of the differential Galois group of NVEs associated with Γ is Abelian.

Since $\mathcal{G}$ always contains $\mathcal{M}$, the Morales-Ramis non-integrability theorem always yields stronger necessary conditions than the Ziglin criterion.

When applying the Morales-Ramis criterion, our first step is to find a non-equilibrium particular solution, which often lies on an invariant submanifold. Next, we calculate the corresponding VEs and NVEs. If we know that our Hamiltonian system possesses k first integrals in involution, then we can consider VEs on one of their common levels, and the order of NVEs is equal to $s = 2(m - k)$ [18], [19]. In the last step we have to check if the identity component of $\mathcal{G}$ is Abelian, a task where the tools of algorithmic Galois theory (such has the Kovacic algorithm) become useful. In practice, we often check only whether that component is solvable (which is equivalent to check whether the NVEs have Liouvillian solutions), because the system is not integrable when that component is not solvable.

Our main objectives in that field are: (i) to apply the Morales-Ramis theory to various dynamical systems occurring in mechanics and astronomy; (ii) to develop algorithms that carry out effectively all the steps of that theory; (iii) to extend it by making use of non-homogeneous variational equations; (iv) to generalize it to various non-Hamiltonian systems, e.g. for systems with certain tensor invariants; (v) to formulate theorems about partial integrability of dynamical systems and about real integrability (for real dynamical systems) in the framework of the Morales-Ramis theory;

3.3. Mathematical web services

Keywords: *Computer algebra*, *MathML*, *OpenMath*, *Web services*, *communication*, *deductive databases*, *formula databases*.

The general theme of this aspect of our work is to develop tools that make it possible to share mathematical knowledge or algorithms between different software systems running at arbitrary locations on the web.

Most computer algebra systems deal with a lot of non algorithmic knowledge, represented directly in their source code. Typical examples are the values of particular integrals or sums. A very natural idea is to group this knowledge into a database. Unfortunately, common database systems are not capable to support the kind of mathematical manipulations that are needed for an efficient retrieval (doing pattern-matching, taking into account commutativity, etc.). The design and implementation of a suitable database raise some interesting problems at the frontier of computer algebra. We are currently developing a prototype for such a database that is capable of doing some deductions. Part of our prototype could be applied to the general problem of searching through mathematical texts, a problem that we plan to address in the near future.

The computer algebra community recognized more than ten years ago that in order to share knowledge such as the above database on the web, it was first necessary to develop a standard for communicating mathematical objects (via interprocess communication, e-mail, archiving in databases). We actively participated in the definition of such a standard, OpenMath (partly in the course of a European project). We were also involved in the definition of MathML by the World Wide Web Consortium. The availability of these two standards is the first step needed to develop rich mathematical services and new architectures for computer algebra and scientific computation in general enabling a transparent and dynamic access to mathematical components. We are now working towards this goal by experimenting with our mathematical software and emerging technologies (Web Services) and participating in the further development of OpenMath.

4. Application Domains

4.1. Panorama

We have applied our research on linear functional systems to mathematical physics (e.g., computation of quadratic first integral or conservation laws, computation of block-diagonal forms of systems appearing in mathematical physics) and to control theory (e.g., computation of parametrizations or autonomous elements, computation of block-diagonal forms of many systems appearing in control theory, stabilization of linear control systems).

We have also applied our algorithms and programs for computing differential Galois groups to determine necessary conditions for integrability in mechanical modeling and astronomy.

Other applications include biology, where our algorithms for solving recurrence equations have been applied to the steady-state equations of nonhomogeneous Markov chains modeling the evolution of microsatellites in genomes.

5. Software

5.1. Maple package diffalg

Keywords: *Nonlinear differential systems*, *analysis of singular solutions*, *differential algebra*, *differential elimination*, *triangulation-decomposition algorithms*.

Participant: Evelyne Hubert.

The *diffalg* is a library that has been part of the commercial release of Maple since Maple V.5 (with an initial version by F. Boulier) and has been developed up to Maple 7 by E. Hubert. The library implements a triangulation-decomposition algorithm for polynomially nonlinear systems and tools for the analysis of singular solutions. Newer releases are available on the web (http://www.inria.fr/cafe/Evelyne.Hubert/diffalg) together with an extensive set of examples of applications and an efficiency test suite. Recent developments include optimized treatment of parameters, improvements for higher degree equations and generalization to non-commuting derivations.

5.2. Library OreModules

Keywords: *Constructive algebraic analysis*, *Gröbner bases*, *Ore algebras*, *constructive homological algebra*, *control theory*, *linear functional systems*, *mathematical physics*.

Participants: Alban Quadrat [correspondent], Daniel Robertz.

The OREMODULES package of Ore_algebra (Ore_algebra is a part of the commercial release of Maple) is dedicated to the algebraic study of linear systems of functional equations defined over some Ore algebras (e.g., ordinary or partial differential equations, time-delay equations, difference equations) and their applications in mathematical physics (e.g., research of potentials, computations of field equations and conservation laws) and in linear control theory (e.g., parameterizability, flatness, autonomous elements, first integrals of motion, equivalences, controllability, observability, input-output behaviour).

The main novelty of OREMODULES is to combine the recent developments of the Gröbner bases over some Ore algebras (non-commutative polynomial rings) with new algorithms of algebraic analysis in order to effectively check classical properties of module theory (e.g., existence of a non-trivial torsion submodule, torsion-freeness, reflexiveness, projectiveness, stably freeness, freeness), give their system-theoretical interpretations (existence of autonomous elements or successive parametrizations, existence of minimal/injective parametrizations, Bézout equations or generalized inverses) and compute important tools of homological algebra (e.g., (minimal) free resolutions, splitting, extension functor, projective dimension, Hilbert power series).

The main advantage of the language of homological algebra used in [29] carries over the implementations in OREMODULES: up to the choice of the domain of functional operators which occurs in a given system, all algorithms are stated and implemented in sufficient generality such that linear systems defined over the Ore algebras developed in the Maple package of *Ore_algebra* are covered at the same time.

A library of more than 30 examples coming from engineering sciences and mathematical physics (e.g., a two pendulum mounted on a car, a wind tunnel model, stirred tank models, a two reflector antenna, an electric transmission line, linearized Einstein equations, Maxwell equations, linear elasticity, Lie-Poisson structures) illustrates the main functions and applications of OREMODULES.

5.3. Library Stafford

Keywords: *Stafford theorems*, *bases of free modules over the Weyl algebras*, *block-diagonalization and block-triangularization problems*, *computation of flat outputs*, *generators of ideals over the Weyl algebras*.

Participants: Alban Quadrat [correspondent], Daniel Robertz.

The STAFFORD package of OREMODULES contains an implementation of constructive versions of the J. T. Stafford's famous but tricky theorem stating that every ideal over the Weyl algebras $A_n(\mathbb{Q})$ and $B_n(\mathbb{Q})$ of differential operators with polynomial and rational coefficients can be generated by two generators. Based on this implementation and on algorithmic results recently obtained by the authors of this package, two algorithms have been implemented which compute bases of free modules over the Weyl algebras $A_n(\mathbb{Q})$ and $B_n(\mathbb{Q})$.

The development of the STAFFORD package was mainly motivated by the wish to compute flat outputs of flat multidimensional linear systems with varying coefficients, a problem appearing in control theory. However, it has recently played an important role in the decomposition problem of linear systems of partial differential equations with polynomial or rational coefficients. For more details, see Section 6.3.

5.4. Library Morphisms

Keywords: *Factorization and decomposition of linear functional systems*, *algebraic analysis*, *constructive homological algebra*, *quadratic conservation laws*, *quadratic first integrals*.

Participants: Thomas Cluzeau, Alban Quadrat [correspondent].

The forthcoming MORPHISMS package of OREMODULES was developed in order to constructively handle some homological tools such as computations of morphisms between two finitely presented modules over Ore algebras, compute kernel, coimage, image and cokernel of morphisms and projectors of the ring of endomorphisms. Using the packages STAFFORD (see Section 5.3) and QUILLEN-SUSLIN (see Section 5.5), these results allow us to effectively compute factorizations as well as finding decompositions of linear systems over Ore algebras. Hence, using the package MORPHISMS, we can highly simplify the form of linear functional systems appearing in mathematical physics and control theory.

In terms of module theory, the MORPHISMS package gives some methods to test if two modules are isomorphic, if a given module contains a submodule (reducible modules) or if it can be written as the direct sum of two submodules. Applications of the MORPHISMS package to mathematical physics and control theory are illustrated in a library of examples (e.g., computation of Galois symmetries of physical systems, quadratic first integral and conservation laws, decomposition of linear systems coming from mathematical physics and control theory, equivalence of systems).

5.5. Library QuillenSuslin

Keywords: *(weakly) doubly factorization*, *Lin-Bose conjecture*, *Quillen-Suslin theorem*, *block-diagonalization and block-triangularization problems*, *flat multidimensional systems*.

Participants: Anna Fabiańska, Alban Quadrat [correspondent].

The forthcoming QUILLENSUSLIN package of OREMODULES, developed by A. Fabiańska (University of Aachen) with the help of A. Quadrat, contains an implementation of the famous Quillen-Suslin theorem. In particular, this implementation allows us to compute bases of free modules over a commutative polynomial ring with coefficients in a field (mainly $\mathbb{Q}$) and in a principal ideal domain (mainly $\mathbb{Z}$). The development of this package was led by different applications of the Quillen-Suslin theorem in multidimensional systems theory (e.g., computation of flat outputs of flat multidimensional systems, equivalence problems, Lin-Bose conjecture, computation of (weakly) doubly coprime factorizations) and signal processing (e.g., parametrization of synthesis filters). Finally, it has recently played a crucial role in the decomposition problem of linear systems of partial differential equations with constant coefficients (see Section 6.3).

5.6. Library libaldor

Keywords: *Aldor*, *arithmetic*, *data structures*, *standard*.

Participant: Manuel Bronstein.

The LIBALDOR library, under development in the project for several years, became the standard ALDOR library, bundled with the compiler distribution since 2001.

5.7. Library Algebra

Keywords: *Computer algebra*, *commutative algebra*, *linear algebra*, *polynomials*.

Participants: Manuel Bronstein, Marc Moreno-Maza [University of Western Ontario].

The ALGEBRA library, written in collaboration with Marc Moreno-Maza, is now also bundled with the ALDOR compiler, starting with version 1.0.2, which has been released in May 2004. It is intended to become a standard core for computer algebra applications written in ALDOR.

5.8. Library Σ^{it}

Keywords: *Computer algebra*, *difference equations*, *differential equations*, *systems*.

Participant: Manuel Bronstein.

The Σ^{it} library contains our algorithms for solving functional equations. The stand-alone solver powers the the web service at http://www-sop.inria.fr/cafe/Manuel.Bronstein/sumit/bernina_demo.html that has solved third order equations as well since the end of 2004.

6. New Results

6.1. Fast algorithms for linear differential and difference equations

Keywords: *Linear differential and difference equations*, *baby step-giant step*, *binary splitting*, *complexity*, *indefinite and definite summation*, *polynomial and rational solutions*.

Participants: Thomas Cluzeau, Alin Bostan [ALGO project], Frédéric Chyzak [ALGO project], Bruno Salvy [ALGO project].

The search for rational (and thus polynomial) solutions of linear differential and difference equations lies at the heart of several important algorithms in computer algebra.

In [23], we investigate the problem of computing polynomial solutions of linear differential equations. Even for detecting the non-existence of non-zero polynomial solutions of a linear differential equation, there exists no general algorithm in polynomial complexity. However, high degree polynomial solutions are controlled by the differential equation. This allows to represent them with a compact data-structure: a linear recurrence and initial conditions. This compact representation enables us to compute polynomial solutions in quasi-optimal complexity (that is, optimal complexity modulo logarithm terms) using binary splitting and baby step/giant step techniques. The timings of our implementations in Maple and Magma confirm the complexity estimates in practice.

We then address the problem of computing rational solutions. The compact representation of polynomial solutions of linear differential equations also allows us to evaluate, in quasi-optimal complexity, their value and that of their derivatives at a rational or algebraic point. We can also perform divisions by high powers of a monomial which lead directly to an algorithm in quasi-optimal complexity to compute rational solutions of linear differential equations.

This year, we have continued our joint work by looking at the case of difference equations. The calculation of polynomial solutions works in the same way except that the coefficients of these polynomials satisfy a linear recurrence equation in the binomial basis instead of the monomial basis. The generalization of our techniques for computing rational solutions is a little bit more difficult because of the appearance of a new phenomenon called the dispersion. However, binary splitting and baby step/giant step techniques together with the compact representation of the numerator allow us to obtain an algorithm in quasi-optimal complexity. As a direct consequence, we obtained fast versions of Abramov's algorithm computing rational solutions of linear recurrence equations, Gosper's algorithm for indefinite summation, and Zeilberger's algorithm for definite summation. Our algorithms have been implemented in Maple. This work has been published in [12].

6.2. Formal solutions of partial differential systems at a singular point

Keywords: *Pfaffian systems*, *normal crossings singularities*, *regular and irregular singularity*.

Participants: Nicolas Le Roux, Moulay Barkatou [University of Limoges], Evelyne Hubert.

Completely integrable systems are well studied. They are encountered for instance in isomonodromic deformations or in differential geometry. We consider here completely integrable systems with normal crossings singularities. These systems can be rewritten locally in the form

$$(\star) \begin{cases} \frac{\partial Y}{\partial x_1} = x_1^{-(p_1+1)} A_1 Y \\ \vdots \\ \frac{\partial Y}{\partial x_1} = x_n^{-(p_n+1)} A_1 Y \end{cases}$$

The A_i are $m \times m$ matrices with entries in the ring $\mathbb{C}[[x_1, ..., x_n]]$ and the p_i are non negative integers. Y is a $m \times 1$ vector of unknowns with coordinates in an extension of $\mathbb{C}((x_1, ..., x_n))$.

We investigate algorithms for the computation of formal solutions of completely integrable systems with normal crossing singularities. We expect to extend the known algorithms for ordinary differential systems.

As in the ordinary case, there are three kinds of singularities at the origin. Singularities of the *first kind* are those for which $p_i = 0$ for all i. Every system of the first kind is equivalent, by a transformation $Y = TZ$ with $T \in GL_m(\mathbb{C}((x)))$, to a system where the matrices have constant entries [43], [85]. These constant matrices are the monodromy of the system. Singularities are *regular* if the system is equivalent to one of the first kind. All other singularities are called *irregular*. There are algorithms to decide whether a singularity is regular that is based on a criterion by P. Deligne [35] and A. van den Essen [36]. Yet this is not constructive as the algorithm does not produce the transformation. We obtained a dual version [7] of the criterion by A. van den Essen. Our criterion is based on the duality between increasing and decreasing versions of Levelt algorithm and characterizes regularity of a singularity by means of stabilization of a decreasing sequence of lattices at the mth stage. Stabilization at the mth stage, not appearing in van den Essen criterion, is of main interest for algorithmic purposes.

For singularities of the first kind the solution space can be described in terms of the monodromy matrices. More precisely, one can find a basis of solutions [43], [85] which form is

$$T x_1^{C^{(1)}} \cdots x_n^{C^{(n)}},$$

where $T \in M_m(\mathbb{C}[[x_1, ..., x_n]])$ with $\det(T) \neq 0$ and the $C^{(i)}$s are $m \times m$ constant matrices commuting with one another. We obtained [7] from proofs in [43], [85] methods for computing such a basis of solutions. We are going to implement these methods in Maple.

The description of the solution space at regular singularities is then obtained if we have the transformation that reduces the system to a system of the first kind.

For irregular singularities, H. Charrière was the first to describe the solution space in the case where the system only involves two independent variables [26]. She showed that Turritin's form for linear differential system [88] can be carried to this case. Her proof uses the lattice approach initiated by R. Gérard & A.H.M. Levelt [42], [43]. Next the result was extended to the general case independently by A.H.M. Levelt & A. van den Essen [37] and H. Charrière & R. Gérard [27].

Our first goal is to provide an algorithm for reducing the order of the singulatrity at the origin - that is the length of $(p_1, ..., p_n)$ - by linear transformations. Such a reduction process would allow us to decide wether or not the singularity is regular. We have investigated the possibility to extend Moser's algorithm [62], [20] and Levelt's algorithm [20] that work in the ordinary case. We clarified the relationship between those two algorithms and gave a complexity study of both of them [7]. The presentation with lattices by Levelt seems easier to extend. We have shown that it can be extended to the case $n = 2$ [15]. It is also implemented in Maple. We have used parts of the implementation of Moser's algorithm by M. Barkatou [20]. In the case $n = 2$, we are building an algorithm for computing a basis of formal solution for systems with irregular singularities, relying on the rank reduction algorithm previously proposed. This is to be written in a forthcoming paper.

For $n > 2$, the difficulties we encounter are twofold. Firstly, the systems considered are not invariant under transformation $Y = TZ$. Secondly, there is no warranty that the lattices are free. The same problem arose in [37], [27].

6.3. Factorization and decomposition of linear functional systems

Keywords: *Factorization and decomposition of linear functional systems*, *algebraic analysis*, *constructive homological algebra*, *quadratic conservation laws*, *quadratic first integrals*.

Participants: Cluzeau Thomas, Alban Quadrat.

The factorization and decomposition problems of systems of linear ordinary differential or difference equations have long been studied in the mathematical and symbolic computation literatures [21], [81], [82], [30], [55], [56] and in the CAFÉ project [89], [24]. The main novelty of the approach developed in the CAFÉ project is to recast these problems within the algebraic analysis framework and to extend them to linear functional systems defined over Ore algebras. Except from the pioneering works of M. F. Singer [81] and his collaborators, this approach is new in the case of finite-dimensional linear systems over Ore algebras and, to our knowledge, is the first one for general linear functional systems (determined/under-/over-determined). The second novelty is to use the expertise developed in the CAFÉ project on efficient implementations for linear systems over univariate Ore algebras (e.g., see Section 6.1). The last novelty is to apply our new results to different systems coming from mathematical physics and control theory.

Following ideas of M. F. Singer [81], [82], in the publications [13] and [33], we have shown how to compute the morphisms from a left D-module M, finitely presented by a matrix R with entries in an Ore algebra D, to a left D-module N presented by a matrix S. If D is a commutative ring or M and N are two finite-dimensional k-vector spaces (e.g., case of ordinary differential systems or integrable connections), then we can effectively compute the set $\hom_D(M, N)$ of morphisms from M to N. If the previous hypotheses are not fulfilled, then we can only compute the set of morphisms of a given order. A morphism from M to N defines a transformation sending a solution of the system $S\,z = 0$ into a solution of $R\,y = 0$. When $S = R$, the ring $\mathrm{end}_D(M)$ of the endomorphisms of M corresponds to the “Galois symmetries” of the system $R\,y = 0$. Moreover, we have explicitly characterized the kernel, coimage, image and cokernel of a morphism from M to N, a fact that can be used to check the equivalence of the systems $R\,y = 0$ and $S\,z = 0$.

We have proved that the existence of a non-injective endomorphism of a left D-module M, finitely presented by a matrix R with entries in an Ore algebra D, corresponded to a factorization of the form $R = R_1\,R_2$, where R_1 and R_2 are two matrices with entries in D. Using this result, we have shown how the integration of the system $R\,y = 0$ could be done by means of a cascade of integrations. Moreover, we have described how to compute the projectors of $\mathrm{end}_D(M)$ and proved that they allowed us to decompose the system $R\,y = 0$ into two decoupled systems $Q_1\,y_1 = 0$ and $Q_2\,y_2 = 0$, where Q_1 and Q_2 were two matrices with entries in D. Hence, the integration of the system $R\,y = 0$ becomes equivalent to the integrations of the two independent systems $Q_1\,y_1 = 0$ and $Q_2\,y_2 = 0$. Then, under certain conditions on the projectors (freeness), we proved that the system $R\,y = 0$ was equivalent to a block diagonal system. In particular, these conditions always hold in the case of a univariate Ore algebra over a field of coefficients (i.e., ordinary differential/difference systems over the field of rational functions) and in the case of a multivariate commutative Ore algebras due to the Quillen-Suslin theorem (e.g., linear system of partial differential equations with constant coefficients) (see Section 5.5). Moreover, if some rank conditions on the projectors are fulfilled, then, using a result due to J. T. Stafford, we prove that a similar result also holds for the Weyl algebras $A_n(k)$ and $B_n(k)$ over a field k of characteristic 0 (i.e., linear system of partial differential equations with polynomial/rational coefficients over k) (see Section 5.3). Using the recent implementations of the Quillen-Suslin and Stafford theorems in the packages STAFFORD and QUILLENSUSLIN, we give a constructive way to compute the block diagonal decomposition of $R\,y = 0$ when it exists. For all the previous results, no condition on the system $R\,y = 0$ is required such as D-finite, determined, underdetermined, overdetermined, i.e., this approach handles general linear systems over an Ore algebra. In the case of a finite-dimensional determined system over an Ore algebra with rational coefficients, we find again the results obtained in M. Wu’s PhD thesis [89].

The different algorithms have been implemented in the package MORPHISMS (see Section 5.4) of OREMODULES (see Section 5.2). This package is available with a library of examples illustrating its main features and applications in mathematical physics (e.g., symmetries of the Maxwell/Euler/Saint-Venant/Einstein equations, quadratic first integrals of motion or conservation laws (e.g., linear elasticity, electromagnetism, hydrodynamics), decomposition of systems of partial differential equations (e.g., transmission lines, Cauchy-Riemann or

Beltrami equations, acoustic or electromagnetic waves, two-dimensional rotational isentropic flow, incompressible fluid in rotation with a small fluid velocity), Monge parametrizations) and control theory (e.g., decomposition of differential time-delay systems appearing in applications such as a wind tunnel model, a flexible rod, a stirred tank, different tank models, a network model).

6.4. Linear systems over Ore algebras and applications

Keywords: *Constructive algebraic analysis*, *Gröbner bases*, *Ore algebras*, *constructive homological algebra*, *control theory*, *linear functional systems*, *mathematical physics*.

Participants: Alban Quadrat, Daniel Robertz.

In the publications [29] [8], we study the structural properties of under-determined linear systems over Ore algebras. Using the recent development of Gröbner and Janet bases over Ore algebras, we show how to make effective some important concepts of homological algebra (e.g., free resolutions, split exact sequences, duality, extension functor, different dimensions, torsion-free degrees, grades). Using these results, we obtain some effective algorithms which check the different structural properties of under-determined linear systems over Ore algebras by means of the algebraic properties of the underlying modules (e.g., torsion/torsion-free/reflexive/projective/stably free/free modules). In particular, we explain why these properties are related to the possibility to successively parameterize all solutions of a system and its parametrizations. Moreover, we show how these properties generalize the well-known concepts of primeness, developed in the literature of multidimensional systems, to systems with varying coefficients. Then, using a dictionary between the structural properties of under-determined linear systems over Ore algebras and concepts of linear control theory, we show that the previous algorithms allow us to effectively check whether or not a multidimensional linear control system over certain Ore algebras is (weakly, strongly) controllable, observable, parameterizable, flat, π-free... The problem of parameterizing the solutions of multidimensional linear systems has been extensively studied in control theory by the school of M. Fliess and J. C. Willems. Our results allow us to effectively answer that problem for classes of functional systems. The different algorithms obtained in [29] have been implemented by D. Robertz and A. Quadrat in package OREMODULES (see Section 5.2 and [8]) and have been illustrated on more than 30 different systems appearing in the literature. See [29] [8] for more details.

The purpose of [76], [75] [17] is to study the so-called *Monge problem* for linear systems over some classes of Ore algebras. This problem was first studied by J. Hadamard and E. Goursat for systems of partial differential equations in the period 1900-1930. Using the algebraic analysis approach developed in [29], we obtain an algorithm to compute a *Monge parametrization* of all the solutions of an underdetermined linear system over an Ore algebra obtained by gluing the autonomous elements of the system to the parameterizable subsystem. The autonomous elements and the parameterizable subsystem can be computed by means of the algorithms developed in [29]. We point out that a Monge parametrization is more general than the parameterizations used in [29] as it depends on arbitrary functions of a certain number of the independent variables whereas we only consider parameterizations depending on all the independent variables in [29]. Effective algorithms checking the given condition are obtained and implemented in OREMODULES (see Section 5.2 and [8]). In particular, these results are useful if we want to parametrize the solutions of certain systems appearing in mathematical physics and control theory (e.g., tank models, flexible rod models, some linear systems of partial differential equations appearing in the theory of elasticity). See [17] [76] for different examples.

Finally, we have shown in [17] [76] how to use the Monge parametrization in the study of variational problems and optimal control. In particular, we proved that the existence of a Monge parametrization of the system allowed us to transform a variational problem with the system as a differential constraint into a variational problem without differential constraints (which can be solved by means of the Euler-Lagrange equations without Lagrangian multipliers).

6.5. Applications of the Stafford theorems and implementation

Keywords: *Stafford theorems*, *bases of free modules over the Weyl algebras*, *block-diagonalization and block-triangularization problems*, *computation of flat outputs*, *generators of ideals over the Weyl algebras*.

Participants: Alban Quadrat, Daniel Robertz [Aachen University].

It is quite well-known that an analytic time-varying controllable ordinary differential system is *flat* outside some singularities [60]. In [74], we proved that every analytic time-varying controllable linear system was a projection of a flat system and we gave an explicit description of the flat system which projects onto a given controllable one. Moreover, we generalized this result to multidimensional linear systems over Ore algebras. Finally, we proved that every controllable multi-input ordinary differential linear system with polynomial (resp., analytic) coefficients is flat, answering a question asked by K. B. Datta. However, the problem of constructively finding the flat outputs of such a flat system was still open. We have recently studied this problem and solved it using techniques of algebraic analysis and symbolic computation.

A well-known and difficult result due to J. T. Stafford asserts that a stably free left module M over the Weyl algebras $D = A_n(k)$ or $B_n(k)$ of differential operators with polynomial/rational coefficients $-$ where k is a field of characteristic 0 $-$ with $\mathrm{rank}_D(M) \geq 2$ is free. The purpose of [16] [73] is to present a constructive proof of this result as well as an effective algorithm for the computation of bases of M. This algorithm, based on the new constructive proofs [45], [54] of J. T. Stafford's result on the number of generators of left ideals over D, performs Gaussian eliminations on the columns of the formal adjoint of the presentation matrix of M. Moreover, we show that J. T. Stafford's result is a particular case of a more general one asserting that a stably free left D-module M with $\mathrm{rank}_D(M) \geq \mathrm{sr}(R)$ is free, where $\mathrm{sr}(R)$ denotes the stable range of a ring D. This result is constructive if the stability of unimodular vectors with entries in D can be tested. An algorithm which computes the left projective dimension of a general left D-module M defined by means of a finite free resolution is presented. In particular, it allows us to check constructively whether or not the left D-module M is stably free. All these algorithms have recently been implemented in the package STAFFORD (see Section 5.3) based on OREMODULES (see Section 5.2). Hence, we can now effectively compute the two generators of any left ideal over D and a basis of any free left D-module.

These results can be used in order to recognize whether or not a linear system of ordinary/partial differential equations with polynomial/rational coefficients admits an injective parametrization. This problem is interesting in the theory of flat systems [40], [41], [60]. Finally, the solution of the decomposition problem obtained in [13] [33] highly uses the computation of bases of free modules. Hence, the decomposition problem can be constructively solved in the case of systems of partial differential equations with polynomial/rational coefficients using the package STAFFORD.

6.6. Applications of the Quillen-Suslin theorem and implementation

Keywords: *(weakly) doubly factorization*, *Lin-Bose conjecture*, *Quillen-Suslin theorem*, *block-diagonalization and block-triangularization problems*, *differential time-delay systems*, *flat multidimensional systems*.

Participants: Anna Fabiańska [Aachen University], Alban Quadrat.

In 1955, a famous conjecture due to Serre asserted that if we take a row vector v with coefficients in a commutative polynomial ring $A = k[x_1, ..., x_n]$ (k is a field) which admits a right-inverse over A, then v can be embedded into a square unimodular matrix, namely, a matrix with a non-zero constant determinant. This conjecture was proved independently by Quillen and Suslin in 1976 and is nowadays called the Quillen-Suslin theorem. In terms of module theory, Serre's conjecture means that a projective A-module is free and the computation of the unimodular matrix then gives an explicit basis for this module. Constructive proofs of the Quillen-Suslin theorem have been largely studied in the literature of constructive algebra (e.g., see [59]). The purpose of the internship of J. Evers (INRIA Sophia Antipolis, CAFÉ project, 2005) was to start implementing the difficult computations of bases for projective A-modules in OREMODULES. This project has recently been finished by A. Fabiańska (University of Aachen, Germany) within the framework of her PhD thesis supervised by Prof. W. Plesken with the help of A. Quadrat. A package called QUILLENSUSLIN will be soon available (see Section 5.5).

The Quillen-Suslin theorem has many applications in mathematical systems theory and signal processing and, in particular, it allows us to constructively compute flat outputs of flat or π-flat shift-invariant multidimensional linear systems such as time-invariant differential time-delay control systems. Hence, the present work gives the first implementation for the computation of flat outputs of flat shift-invariant multidimensional linear systems [14] [38]. Applications of such a class of systems were largely demonstrated in the literature of control theory and, in particular, for the motion planning and tracking problems (e.g., see the library of examples of OREMODULES and the references therein).

Moreover, using a constructive version of the Quillen-Suslin theorem, A. Fabiańska and A. Quadrat show in [14] [38] that every flat shift-invariant multidimensional linear system is equivalent to a controllable 1-D linear system obtained by setting to 0 all but one functional operators in the system matrix. It was also proved that a flat differential time-delay linear system is equivalent to the controllable ordinary differential linear system without delays, i.e., the system obtained by setting the time-delay amplitude to 0. The implementation of the Quillen-Suslin theorem gives a constructive way to get the precise invertible transforms between these systems. Applications of these results in the study of stabilization problems were started in [14] but will be developed in forthcoming publications. Finally, other applications of the Quillen-Suslin theorem were demonstrated in [14] [38] and, in particular, we give constructive algorithms for proving the standard Lin-Bose's conjecture [57] (generalization of the Serre conjecture), computing (weakly) doubly coprime factorizations of transfer matrices of multidimensional systems and computing bases for the decomposition problems (key point for this problem; see Section 6.3 for more details). These different algorithms have been implemented in QUILLENSUSLIN.

6.7. Resolvent representation for systems of ordinary differential equations

Keywords: *Differential algebra*, *resolvent representation*, *triangulation-decomposition algorithms*.

Participants: Thomas Cluzeau, Evelyne Hubert.

We had previously shown [31] the existence of resolvent representations for systems of (nonlinear) ordinary differential equations that define regular differential ideals. This representation is based on the fact that those systems are birationally equivalent to a single equation. The resolvent representation generalizes the representation used in geometric resolution which has led to probabilistic algorithms of bounded complexity for solving polynomial systems. In [32] we provide practical algorithms for computing such representations. We propose two different approaches. The first one uses differential characteristic decompositions whereas the second one proceeds by prolongation and algebraic elimination. Both constructions depend on the choice of a tuple over the differential base field and their success relies on the chosen tuple to be separating. The probabilistic aspect of the algorithms comes from this choice. To control it, we exhibit a family of tuples for which we can bound the probability that one of its element is separating.

6.8. Algebraic and differential Invariants

Keywords: *Algebraic invariants*, *Lie groups*, *algebraic groups*, *cross-section*, *differential invariants*.

Participants: Evelyne Hubert, Irina Kogan [North Carolina State University], Alexandre Sedoglavic [University of Lille], Eric Schost [Ecole Polytechnique, University of Western Ontario].

In [9] we proposed an algorithm to compute a generating set of rational invariants for the rational action of an algebraic group. The algorithms are implemented in Maple (http://www.inria.fr/cafe/Evelyne.Hubert/aida). In [47] we showed the connection of this algebraic construction with the moving frame method of Fels & Olver [39]. In particular we exhibit a family of algebraic invariants, i.e. algebraic functions of the rational invariants, that not only generate algebraically all algebraic invariants but also generate, functionally, the smooth invariants in the neighbourhood of a point whose orbit is of maximal dimension. Those algebraic invariants occur naturally in differential geometry, as for instance the (Euclidean) curvature.

An essential property of the algebraic invariants we consider is that the field they generate is isomorphic to the field of rational functions on a given cross-section to the orbits. Their algebraic structure is therefore simple and is the basis for determining the differential algebraic structure of the differential invariants. We have made some progress on the determination of this differential algebra.

A particularly favorable case is when the cross-section intersects generic orbits only once, in which case we say the cross-section is of degree one. Such cross-sections are known to exist for *special* groups [67], which include the general linear group and the connected solvable groups. The joint project with E. Schost has been to provide a construction for those cross-sections of degree 1.

Invariants are the natural objects to translate algebraically the geometry and symmetry of problems in diverse fields of science. They allow to reduce the dimension of problems by factoring out the symmetry. We applied those ideas for the nondimensionalisation of dynamical systems. Reducing the number of parameters in biological models retaining the qualitative properties of the dynamics has often been considered an art [63]. One can observe though that the classical reductions performed correspond to the symmetry reduction for simple scalings or translations. We provided the algorithms to determine such symmetries, compute the invariants and the rewriting algorithms to perform the reductions. The whole problems boils down to linear algebra and the algorithms are shown to be polynomial time [48]. An implementation in Maple applied to a library of examples from the mathematical biology litterature has been constructed.

6.9. Algebraic analysis approach to infinite-dimensional linear systems

Keywords: *Infinite-dimensional systems*, *algebraic analysis*, *analysis and synthesis problems*, *internal/strong/simultaneous/robust/optimal stabilization*, *parametrization of all the stabilizing controllers*.

Participant: Alban Quadrat.

Until today, no one has seemed to be able to algebraically incorporate the initial and limit conditions of a system of partial differential equations within the differential module approach (D-modules). From analytical and engineering points of view, the algebraic D-module approach is not yet totally satisfactory. Driven by applied questions arising in control theory, A. Quadrat has tried to develop a new algebraic analysis approach which could solve this negative point for a class of linear systems of partial differential equations depending on two or three independent variables (typically, one independent variable is the time variable and one or two independent variables are the space variables) usually encountered in engineering sciences. This class of systems is usually called *infinite-dimensional linear systems* [34] as they have infinite-dimensional state-spaces or, equivalently, a certain number of functions need to be prescribed as initial or limit conditions in order to get a functional solution. Examples of infinite-dimensional systems are the systems of partial differential equations (e.g., wave, heat, Euler-Bernoulli, telegraphist equations), differential time-delay systems (e.g., transport equation, transmission lines, electrical lines) and fractional differential systems. Therefore, we have tried to develop a new algebraic analysis for infinite-dimensional linear systems and apply it to classical problems of control theory.

Using *symbolic analysis*, also called *symbolic calculus*, developed by O. Heaviside, P. Dirac and L. Schwartz, i.e., techniques based on Laplace transform, convolution products, distributions and symbolic integration (special functions), we can transform an infinite-dimensional linear system into a transfer matrix, i.e., into functional relations between the state and the outputs of the system and its inputs and initial conditions. The transfer matrix carries important information of the system and particularly its stability (e.g., unstable poles correspond to unstable modes). Combining natural functional algebras, which were introduced for studying the stability of the systems, and the fractional representation approach to systems [87] given by the symbolic analysis approach, we have shown in the publications [68], [69], [70], [71], [72] how to develop a new module-theoretic approach to infinite-dimensional linear systems.

Using the algebraic concepts of lattices, we have shown in the publications [10], [11], how to generalize the well-known Youla-Kučera parametrization of all stabilizing controllers for multi-input multi-ouput systems which do not necessarily admit coprime factorizations. The Youla-Kučera parametrization has been the cornerstone of the success of the H_∞-control in the recent years as this parametrization allows us to rewrite the problem of finding the optimal stabilizing controllers (for a certain norm) as affine, and thus, convex problems.

Finally, the algebraic analysis techniques also allowed us to positively solve in the publication [11] the well-known conjecture of Z. Lin [58] in the literature of multidimensional systems on the equivalence between internal stabilizability and the existence of doubly coprime factorizations for multidimensional systems.

7. Other Grants and Activities

7.1. European initiatives

7.1.1. PAI Amadeus

A collaboration with RISC (Research Institute in Symbolic Computation) in Hagenberg, Austria was funded by the PAI program AMADEUS. The coordinators are Ralf Hemmecke for the Austrian side and Evelyne Hubert replaced Manuel Bronstein in June 2005 for the French side. One objective of the collaboration is to develop and implement a very generic and efficient engine to compute Gröbner basis with ideals and modules over rings of linear operators, such as Ore algebras or Poincare-Birkhoff-de Witt algebras. This collaboration ends this year.

7.1.2. PAI Procope

The purpose of this collaboration of A. Quadrat with the Lehrstuhl B für Mathematik RWTH - Aachen, led by Prof. W. Plesken, is to implement new symbolic packages for the mathematical study of systems theory (e.g., the STAFFORD, QUILLEN-SUSLIN, MORPHISMS packages of OREMODULES) and develop their applications to control theory and mathematical physics. This collaboration will continue in 2007.

7.2. Visiting scientists

7.2.1. France

Frédéric Nataf, Laboratoire J.L. Lions, University Paris 6, visited the CAFÉ project to give a talk on the use of the Smith forms in decomposition domains and perfectly matched layers and to work with Alban Quadrat on the Smith and Jacobson forms (16-17/01).

Eric Schost, Ecole Polytechnique, visited the CAFÉ project to collaborate with E. Hubert (18-20/04).

Jean-François Pommaret, Ecole Nationale des Ponts et Chaussées (CERMICS), visited the CAFÉ project for collaborating with A. Quadrat (25-29/09).

7.2.2. Europe

Within the PAI Procope, A. Fabiańska, University of Aachen, visited A. Quadrat from April 2nd to 7th and from November 20th to 24th. Moreover, D. Robertz, University of Aachen, will visit us in January (07-12/01/07) (see Section 7.1).

Ralf Hemmecke (RISC Linz, Austria) visited our project for one week in July (10-15) in order to collaborate with Alban Quadrat on the design of the LIBALDOR and ALGEBRA libraries (see Scetion 5.6) to include non-commutative polynomials such as the ones developed in OREMODULES. The visit is funded by the PAI Amadeus (see Section 7.1).

8. Dissemination

8.1. Leadership within scientific community

- Evelyne Hubert served as a referee for the PhD thesis of L. D'Alfonso, University of Buenos Aires, Argentina.

- Evelyne Hubert is member of the council of the association *femmes & mathématiques*. As such she participates to the monthly meetings of the council to propose, discuss and undertake the actions of the association. Those aim at promoting scientific education and careers for women.
- Evelyne Hubert is a member of the visiting scientist committee chaired by Denis Talay.
- Alban Quadrat has just been elected Associate editor of the international journal Multidimensional Systems and Signal Processing (http://www.springerlink.com/content/1573-0824/) (Springer) for the period 2007-2010.

8.2. Teaching

- Evelyne Hubert taught a course of practical computer algebra in ISIA (April 2006), a Master affiliated to the *Ecole des Mines de Paris*.
- Alban Quadrat gave four lectures at the *Mathematics, Algorithms and Proofs (MAP) workshop* (4 hours), Castro Urdiales, Spain (9-13/01) (http://www.disi.unige.it/map/).
- Alban Quadrat gave two lectures (4 hours) on the factorization and decomposition problems of functional linear systems at the Institut de Recherche en Communications et en Cybernétique de Nantes (http://www.irccyn.ec-nantes.fr/) (28-30/11).
- Alban Quadrat gave two lectures (4 hours) on the factorization and decomposition problems of functional linear systems at the Lehrstuhl B für Mathematik, Aachen University (http://wwwb.math.rwth-aachen.de/index_en.html) (18-20/12).

8.3. Dissertations and internships

Doctorates completed in 2006:

1. Nicolas Le Roux, Université de Limoges : *Solutions Formelles d'Equations aux dérivées partielles*. Co-supervised by Moulay Barkatou (University of Limoges) and Evelyne Hubert.

Doctorates in progress in the project:

1. Anna Fabiańska is partially supervised by A. Quadrat in the framework with her PhD thesis at the University of Aachen (Germany) with Prof. W Plesken.

8.4. Conferences and workshops, invited conferences

T. Cluzeau

- participated to *Journées Matrice Structurées*, University of Limoges, France (18-19/01) (http://www.unilim.fr/pages_perso/jacques-arthur.weil/CF/matstruct.html).
- gave a talk at the conference *International Symposium on Symbolic and Algebraic Computations - ISSAC 2006*, (http://issac2006.dima.unige.it/), Genova, Italy (July 09-12).
- presented his results at the seminar *Laboratoire d'Arithmétique, de Calcul Formel et d'Optimisation*, (http://www.unilim.fr/laco/seminaires/calcul/index.html), University of Limoges, France, (07/04).
- gave a talk at the seminar at the *Institut d'Informatique et Mathématiques Appliquées de Grenoble (IMAG)*, Laboratoire de Modélisation et Calcul, Grenoble, France, (13/04).

E. Hubert

- talked at the seminar of the SALSA project-team in Paris, France (February 14th).
- was invited to present her work at the conference *Gröbner bases in Symbolic Analysis* http://www.ricam.oeaw.ac.at/srs/groeb at the Radon Institute for Computational and Applied Mathematics in Linz, Austria (May 8-12).
- participated to the conference *Théorie de l'Elimination* http://www-sop.inria.fr/galaad/conf/jouanolou/ at the Centre International de Recherche en Mathématiques, Luminy, France (May 15-19).
- gave a talk at the seminar *Calcul formel et Complexité* at the Institut de Recherche Mathématique de Rennes, France (June 19th).
- contributed to a panel session at the conference *Challenges in Computer Algebra Software* (http://www.dagstuhl.de/06271) that took place in Dagstuhl, Germany (July 2-7).
- was invited to present her work at the occasion of the IMA summer program *Overdetermined systems of Partial Differential Equations and Symmetries* (http://www.ima.umn.edu/2005-2006/SP7.17-8.4.06) in Minneapolis, USA (July 17th - August 4th).
- presented a poster at the workshop *Algorithms in Algebraic Geometry* (http://www.ima.umn.edu/2006-2007) organized at the Institute For mathematics and its Applications, Minneapolis, USA (September 15th - 20th).
- was invited to give a talk at the workshop *Software for Algebraic Geometry* (http://www.ima.umn.edu/2006-2007) organized at the Institute For Mathematics and its Applications, Minneapolis, USA (October 22th - 27th).
- was an invited speaker at the workshop *Global Integrability of Field Theories* (http://www.gift2006.eu) that took place at the Cockcroft Institute in Daresbury, UK, (November 1st - 3rd).
- participated to the workshop on *Mathematics in the Digital Age* (http://www.ima.umn.edu/2006-2007) organized at the Institute For mathematics and its Applications, Minneapolis, USA (December 7th - 9th).
- participated to the Kolchin Seminar (http://www.sci.ccny.cuny.edu/~ksda) in New York, USA (December 15th - 16th): she gave a tutorial talk at the Graduate Center series and presented her recent work at the Hunter series.

A. Quadrat

- was invited to give four lectures at the *Mathematics, Algorithms and Proofs (MAP) workshop* (4 hours), Castro Urdiales, Spain (9-13/01) (http://www.disi.unige.it/map/).
- was invited to two talks at the conference *Gröbner bases in Symbolic Analysis* (http://www.ricam.oeaw.ac.at/srs/groeb) at the Radon Institute for Computational and Applied Mathematics in Linz, Austria (8-12/05) (workshops "Gröbner Bases in Symbolic Analysis" and "Gröbner Bases, Control Theory and Signal Processing").
- was invited to give a talk at the seminar of the Algorithms project, INRIA Rocquencourt, (26/05) (http://algo.inria.fr/seminars/).
- organized a mini-symposium entitled "Symbolic methods in multidimensional systems theory" at the *17 th International Symposium on Mathematical Theory of Networks and Systems (MTNS2006)*, Kyoto Japan, (24-28/07) (http://www-ics.acs.i.kyoto-u.ac.jp/mtns06/).

The CAFÉ project

- organized the conference *CAFE, Computer Algebra and Functional Equations. An international conference, in memory of Manuel Bronstein*, Sophia Antipolis, France, (13/07) (http://www-sop.inria.fr/cafe/Cafe2006eng.html).

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