INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Project-Team VEGAS

Effective Geometric Algorithms for Visibility and Surfaces

Nancy - Grand Est

THEME SYM

Activity Report

2008

Table of contents

1. Team

Research Scientist

Sylvain Lazard [Team Leader, Research scientist (CR) INRIA, HdR]
Sylvain Petitjean [Vice Team Leader, Research scientist (DR) INRIA, HdR]
Xavier Goaoc [Research scientist (CR)]
Marc Pouget [Research scientist (CR)]

Faculty Member

Laurent Dupont [Assistant professor, Université Nancy 2]
Hazel Everett [Professor, Université Nancy 2, HdR]

Technical Staff

Julien Clement [until Sept.]

PhD Student

Guillaume Batog [MENESR, since Sept.]
Julien Demouth [MENESR, until Sept.]
Luis Penaranda [Bourse CORDI, since Oct. 2006]
Maria Pentcheva [MENESR, since Oct. 2004]
Linqiao Zhang [INRIA/McGill University, since Jan. 2004]

Post-Doctoral Fellow

Jinsan Cheng
Yi Jun Yang [from Nov.]

Administrative Assistant

Antoinette Courrier [Research technician (TR) CNRS, until Mar.]
Sylvie Thomas [Research technician (TR) CNRS, from Apr.]

Other

Christian Gillot [Master Nancy, Mar.–Jun.]
Guillaume Batog [Master Paris, Mar.–Sept.]
Stephan Koenig [Master Internship INRIA TU Munich, Mar.–Sept.]

2. Overall Objectives

2.1. Introduction

VEGAS is a research project of LORIA (Lorraine Research Laboratory in Computer Science and Applications), a laboratory shared by INRIA (National Institute for Research in Computer Science and Control), CNRS (National Center for Scientific Research), Université Henri Poincaré Nancy 1, Université Nancy 2, and INPL (National Engineering Institute of Lorraine).

The main scientific objective of the VEGAS research team is to *contribute to the development of an effective geometric computing* dedicated to *non-trivial geometric objects*. Included among its main tasks are the study and development of new algorithms for the manipulation of geometric objects, the experimentation of algorithms, the production of high-quality software, and the application of such algorithms and implementations to research domains that deal with a large amount of geometric data, notably solid modeling and computer graphics.

Computational geometry has traditionally treated linear objects like line segments and polygons in the plane, and point sets and polytopes in three-dimensional space, occasionally (and more recently) venturing into the world of non-linear curves such as circles and ellipses. The methodological experience and the know-how accumulated over the last thirty years have been enormous.

For many applications, particularly in the fields of computer graphics and solid modeling, it is necessary to manipulate more general objects such as curves and surfaces given in either implicit or parametric form. Typically such objects are handled by approximating them by simple objects such as triangles. This approach is extremely important and it has been used in almost all of the usable software existing in industry today. It does, however, have some disadvantages. Using a tessellated form in place of its exact geometry may introduce spurious numerical errors (the famous gap between the wing and the body of the aircraft), not to mention that thousands if not hundreds of thousands of triangles could be needed to adequately represent the object. Moreover, the curved objects that we consider are not necessarily everyday three-dimensional objects, but also abstract mathematical objects that are not linear, that may live in high-dimensional space, and whose geometry we do not control. For example, the set of lines in 3D (at the core of visibility issues) that are tangent to three polyhedra span a piecewise ruled quadratic surface and the lines tangent to a sphere correspond, in projective five-dimensional space, to the intersection of two quadratic hypersurfaces.

Effectiveness is a key word of our research project. By requiring our algorithms to be effective, we imply that the algorithms should be *robust*, *efficient*, and *versatile*. By robust we mean algorithms that do not crash on degenerate inputs and always output topologically consistent data. By efficient we mean algorithms that run reasonably quickly on realistic data where performance is ascertained both experimentally and theoretically. Finally, by versatile we mean algorithms that work for classes of objects that are general enough to cover realistic situations and that account for the *exact geometry* of the objects, in particular when they are curved.

2.2. Highlights of the year

We developed, over the last years, the first efficient algorithm for computing an exact parameterization of the intersection of two quadric surfaces (ellipsoids, paraboloids, hyperboloids, etc.) in three-dimensional real space given by implicit equations with rational coefficients. We also produced a C++ implementation of this algorithm which is the only existing solution for computing exactly a parameterization of the intersection of two quadrics. A series of three articles describing the theoretical contributions of this work was published this year in the *Journal of Symbolic Computation* [17], [18], [19].

We proved an important folklore theorem in computer graphics that a polyhedron of complexity n that approximates a surface in some reasonable way has silhouettes of expected size $O(\sqrt{n})$, where the average is taken over all points of view. This confirms a widely accepted belief in computer graphics and is the first complexity result for silhouettes of non-convex polyhedra. This result was published this year in the journal *Discrete and Computational Geometry* [21].

We organized in March 2008 at INRIA the main European conference on computational geometry, the 24th European Workshop on Computational Geometry (EuroCG'08). This drew 140 participants from 19 countries.

3. Scientific Foundations

3.1. Theory and applications of three-dimensional visibility

Keywords: *3D visibility*, *effective geometry*, *robustness*.

The notion of 3D visibility plays a fundamental role in computer graphics. In this field, the determination of objects visible from a given point, the extraction of shadows or of penumbra boundaries are examples of visibility computations. In global illumination methods (e.g. radiosity algorithms), it is necessary to determine, in a very repetitive manner, if two points of a scene are mutually visible. The computations can be excessively expensive. In radiosity, it is not unusual that 50 to 70% of the simulation time is spent answering visibility queries.

Objects that are far apart may have very complicated and unintuitive visual interactions, and because of this, visibility queries are intrinsically global. This partially explains that, until now, researchers have primarily used ad hoc structures, of limited scope, to answer specific queries on-the-fly. Unfortunately, experience has shown that these structures do not scale up. The lack of a well-defined mathematical foundation and the non-exploitation of the intrinsic properties of 3D visibility result in structures that are not usable on models consisting of many hundreds of thousands of primitives, both from the viewpoint of complexity and robustness (geometric degeneracies, aligned surfaces, etc.).

We have chosen a different approach which consists of computing ahead of time (that is, off-line) a 3D global visibility structure for which queries can be answered very efficiently on-the-fly (on line). The 3D visibility complex – essentially a partition of ray space according to visibility – is such a structure, recently introduced in computational geometry and graphics [53], [59]. We approach 3D global visibility problems from two directions: we study, on the one hand, the theoretical foundations and, on the other hand, we work on the practical aspects related to the development of efficient and robust visibility algorithms.

From a theoretical point of view, we study, for example, the problem of computing lines tangent to four among k polytopes. We have shown much better bounds on the number of these tangents than were previously known [45]. These results give a measure of the complexity of the vertices (cells of dimension 0) of the visibility complex of faceted objects, in particular, for triangulated scenes.

From a practical point of view, we have, for example, studied the problem of the complexity for these 3D global visibility structures, considered by many to be prohibitive. The size of these structures in the worst case is $O(n^4)$, where n is the number of objects in the scene. But we have, in fact, shown that when the objects are uniformly distributed, the complexity is linear in the size of the input [6]. This probabilistic result does not prejudice the complexity observed in real scenes where the objects are not uniformly distributed. However, initial empirical studies show that, even for real scenes, the observed complexity is largely inferior to the theoretical worst-case complexity, as our probabilistic result appears to indicate.

We are currently working on translating these positive signs into efficient algorithms. We are studying new algorithms for the construction of the visibility complex, putting the accent on the complexity and the robustness.

3.2. Reliable geometric computations on surfaces

Keywords: *Effective geometry*, *model conversion*, *modeling*, *robustness*.

Simple algebraic surfaces cover a variety of forms sufficient for representing the majority of objects encountered in the fields of design, architecture and industrial manufacturing. For instance, it has been estimated that 95% of all mechanical pieces can be well modeled by quadric patches (degree 2 surfaces, including planes, spheres, cylinders and cones) and torii [60]. It is important, then, to be able to process these surfaces in a robust and efficient manner, notably in view of their use in realistic rendering.

In comparison with polygonal representations, modeling and visualization of scenes of quadrics pose new problems. We study, in particular, problems related to the visualization and realistic rendering of such models. We work alongside the members of the ALICE team on the development of the method called virtual meshing which allows us to go beyond the real geometry of objects by creating a geometric abstraction better adapted to light calculations [41].

Early in the rendering process, but along with the development of a tool for illuminating curved surfaces, it is important to have a reliable conversion process from volumetric to surface models. Many conventional modelers are based on the assembly – union, intersection, difference – of simple volumes (a paradigm called Constructive Solid Geometry or CSG), typically quadric volumes. On the other hand, illumination by the radiosity method can only be done on surface representations of objects (called BRep for Boundary Representation). It is necessary, therefore, to be able to pass, in a robust manner, from one representation to the other, an operation known as CSG-BRep conversion, in order to profit from the power of the virtual mesh [41]. The idea is to make the geometric information coherent with the topological information that expresses the relations of proximity and inclusion of different elements.

A fundamental step of this conversion is the computation of the intersection of two primitive volumes. We have recently developed and implemented a robust and near-optimal algorithm for the computation of an exact parametric form of the intersection of two quadrics [9], [17], [18], [19]. Our method is based on the projective formalism, techniques of linear algebra and number theory, and new theorems characterizing the rationality of the intersection. This is the first general approach to the intersection of two quadrics that is usable in practice (as opposed to the approach used until now, that of J. Levin [58]).

Lately we have worked on the use of this general algorithm in an application context. We continue to work on the development of a loop for exact CSG-BRep conversion for models in which the basic primitives are quadric volumes. This work calls for the resolution of algebraic systems for which we collaborate with the members of the SALSA project.

4. Application Domains

4.1. Computer graphics

Our main application domain is photorealistic rendering in computer graphics. We are especially interested in the application of our work to virtual prototyping, which refers to the many steps required for the creation of a realistic virtual representation from a CAD/CAM model.

When designing an automobile, detailed physical mockups of the interior are built to study the design and evaluate human factors and ergonomic issues. These hand-made prototypes are costly, time consuming, and difficult to modify. To shorten the design cycle and improve interactivity and reliability, realistic rendering and immersive virtual reality provide an effective alternative. A virtual prototype can replace a physical mockup for the analysis of such design aspects as visibility of instruments and mirrors, reachability and accessibility, and aesthetics and appeal.

Virtual prototyping encompasses most of our work on effective geometric computing. In particular, our work on 3D visibility should have fruitful applications in this domain. As already explained, meshing objects of the scene along the main discontinuities of the visibility function can have a dramatic impact on the realism of the simulations.

4.2. Solid modeling

Solid modeling, i.e., the computer representation and manipulation of 3D shapes, has historically developed somewhat in parallel to computational geometry. Both communities are concerned with geometric algorithms and deal with many of the same issues. But while the computational geometry community has been mathematically inclined and essentially concerned with linear objects, solid modeling has traditionally had closer ties to industry and has been more concerned with curved surfaces.

Clearly, there is considerable potential for interaction between the two fields. Standing somewhere in the middle, our project has a lot to offer. Among the geometric questions related to solid modeling that are of interest to us, let us mention: the description of geometric shapes, the representation of solids, the conversion between different representations, data structures for graphical rendering of models and robustness of geometric computations.

4.3. Fast prototyping

We work in collaboration with CIRTES on rapid prototyping. CIRTES, a company based in Saint-Dié-des-Vosges, has designed a technique called Stratoconception® where a prototype of a 3D computer model is constructed by first decomposing the model into layers and then manufacturing separately each layer, typically out of wood of standard thickness (e.g. 1 cm), with a three-axis CNC (Computer Numerical Controls) milling machine. The layers are then assembled together to form the object. The Stratoconception® technique is cheap and allows fast prototyping of large models.

When the model is complex, for example an art sculpture, some parts of the models may be inaccessible to the milling machine. These inaccessible regions are sanded out by hand in a post-processing phase. This phase is very consuming in time and resources. We work on minimizing the amount of work to be done in this last phase by improving the algorithmic techniques for decomposing the model into layers, that is, finding a direction of slicing and a position of the first layer [57].

5. Software

5.1. QI: Quadrics Intersection

Participants: Laurent Dupont, Sylvain Lazard, Sylvain Petitjean.

QI stands for "Quadrics Intersection". QI is the first exact, robust, efficient and usable implementation of an algorithm for parameterizing the intersection of two arbitrary quadrics, given in implicit form, with integer coefficients. This implementation is based on the parameterization method described in [9], [17], [18], [19] and represents the first complete and robust solution to what is perhaps the most basic problem of solid modeling by implicit curved surfaces.

QI is written in C++ and builds upon the LiDIA computational number theory library [39] bundled with the GMP multi-precision integer arithmetic [38]. QI can routinely compute parameterizations of quadrics having coefficients with up to 50 digits in less than 100 milliseconds on an average PC; see [9] for detailed benchmarks.

Our implementation consists of roughly 18,000 lines of source code. QI has being registered at the Agence pour la Protection des Programmes (APP). It is distributed under the free for non-commercial use INRIA license and will be distributed under the QPL license in the next release. The implementation can also be queried via a web interface [40].

Since its official first release in June 2004, QI has been downloaded six times a month on average and it has been included in the geometric library EXACUS developed at the Max-Planck-Institut für Informatik (Saarbrücken, Germany). QI is also used in a broad range of applications; for instance, it is used in photochemistry for studying the interactions between potential energy surfaces, in computer vision for computing the image of conics seen by a catadioptric camera with a paraboloidal mirror, and in mathematics for computing flows of hypersurfaces of revolution based on constant-volume average curvature.

5.2. Isotop: Topology and Geometry of Planar Algebraic Curves

Participants: Jinsan Chen, Sylvain Lazard, Luis Penaranda, Marc Pouget.

ISOTOP is a Maple software for computing the topology of an algebraic plane curve, that is, for computing an arrangement of polylines isotopic to the input curve. This problem is a necessary key step for computing arrangements of algebraic curves and has also applications for curve plotting. This software was developed over 2007 and 2008 in collaboration with E. Tsigaridas from INRIA Sophia Antipolis - Méditerranée (GALAAD) and F. Rouillier from INRIA Paris - Rocquencourt (SALSA). It is based on the method described in [23] which incorporates several improvements over previous methods. In particular, our approach does not require generic position (nor shearing) and avoids the computations of sub-resultant sequences (in all cases). Our preliminary implementation is competitive with other implementations (such as ALCIX and INSULATE developed at MPII Saarbrücken, Germany and TOP developed at Santander Univ., Spain). It performs similarly for small-degree curves and performs significantly better for higher degrees, in particular when the curves are not in generic position.

5.3. CGAL: Computational Geometry Algorithms Library

Participants: Marc Pouget, Luis Penaranda.

Born as a European project, CGAL (http://www.cgal.org) has become the standard library for computational geometry. It offers easy access to efficient and reliable geometric algorithms in the form of a C++ library. CGAL is used in various areas needing geometric computation, such as: computer graphics, scientific visualization, computer aided design and modeling, geographic information systems, molecular biology, medical imaging, robotics and motion planning, mesh generation, numerical methods...

M. Pouget is co-author and maintainer, with F. Cazals from the Geometrica team, of two packages released in the version (3.3) of the library. These packages belong to the geometry processing part, they enable the *Approximation of Ridges and Umbilics* [47] and the *Estimation of Local Differential Properties* [48] on triangulated surface meshes. The algorithm for the estimation of differential properties also appeared in the ACM Transactions on Mathematical Software [14].

L. Penaranda should submit in 2008 a package which is a model of the *Univariate Algebraic Kernel* concept for algebraic computations. It enables the exact computation of arrangement of polynomial functions [28]. This implementation uses the RS library developed by Fabrice Rouillier at INRIA Paris - Rocquencourt (SALSA) for isolating real roots of polynomials.

6. New Results

6.1. Effective 3D global visibility, theory and applications

Participants: Julien Demouth, Hazel Everett, Xavier Goaoc, Sylvain Lazard, Sylvain Petitjean, Linqiao Zhang.

When an observer moves in a 3D scene, his view of the scene changes qualitatively when he goes through certain critical positions. This idea underlies many approaches to visibility questions in Computer Vision, Computer Graphics and Computational Geometry. The locus of critical points, or *visual-event surfaces*, depends on the precise meaning attached to “view” and “qualitative change”. The two traditional approaches to these questions consider objects that are either smooth manifolds or polyhedra, and capture different phenomena; in particular, the visual-event surfaces of a smooth manifold cannot be easily retrieved from the visual-event surfaces of a sequence of polyhedral approximations of this manifold. We studied the visual-event surfaces associated with the topological changes in the apparent contour of an object, a notion that is well-defined both in the smooth and discrete setting. For scenes consisting of disjoint convex sets, we characterized these visual-event surfaces and showed that this notion captures the essential properties of the previous smooth and discrete approaches, while being simpler to analyze [11]. In the case of polyhedral manifolds, we gave a partial characterization of these visual-event surfaces [35].

Related to visual-event surfaces, shadows play a central role in human perception and a wide variety of approaches have been considered for simulating and rendering them. Unfortunately, computing realistic shadows efficiently is a very difficult problem, particularly in the case of non-point light sources, due to the complicated internal structure that such shadows may have. We presented some surprising combinatorial results on the umbra and penumbra cast by non-trivial light sources. A point is in the umbra if it does not see any part of any light source; it is in full light if it sees entirely all the light sources; otherwise, it is in the penumbra. For instance, we proved that a segment light source may cast on a plane, in the presence of two fat polytopes of size n, up to $\Theta(n)$ connected components of umbra in the worst case. We also presented several lower and upper bounds in the case of segment and polygonal light sources in a scene composed of polytopes. These results are the first non-trivial bounds on the size of the umbra and show that the umbra can be surprisingly complicated, even in the presence of disjoint fat obstacles. These results were accepted this year to the journal *Computational Geometry, Theory and Applications* [16] in a special issue of selected papers from the 23rd European Conference on Computational Geometry in 2007. More generally, this work gives some insight on the structure of the umbra. Combined with the insight obtained on visual-event surfaces, we developed a new method for computing the exact boundaries between the full light, penumbra and umbra regions of first-order shadows cast by area light-sources; the associated prototype software is, up to our knowledge, the first implementation that computes exactly such shadow boundaries of polygonal light sources in the presence of polyhedral obstacles for nontrivial scenes (*e.g.,* 50 polytopes with 1 500 vertices) [11]. Many of these results were achieved in the context the Ph.D. thesis of Julien Demouth which was defended in November [11].

Lines tangent to geometric objects play a fundamental role in visibility problems, for instance, as generators of shadow boundaries. We studied over the last years the complexity of the set of lines and free line segments tangent to possibly intersecting arbitrary convex polytopes and presented an efficient sweep-plane algorithm for computing such lines or free segments [45]. We recently completed a first implementation of this algorithm that computes robustly the vertices of the visibility skeleton of a set of convex polyhedra in generic position [61]. This implementation is a key element of the prototype described above for computing limits of umbra; it also allowed us to study experimentally the size of the 3D visibility skeleton (a data structures that encodes visibility information) in a random setting and, in particular, to show that the constant involved in the asymptotic complexity is small [30]. Related to this work, we studied various predicates, arising in three-dimensional visibility, concerning line transversals to lines and segments in 3D. In particular, we computed the degrees of standard methods of evaluating these predicates. We showed that the degrees of some of these methods are surprisingly high (up to 168), which may explain why computing line transversals with finite-precision floating-point arithmetic is prone to error. This work was accepted this year in the journal *Computational Geometry, Theory and Applications* [20]. Due to the high degree of these standard predicates, we explored alternatives and presented efficient predicates for solving such elementary queries [26] which are important for the development of robust and efficient algorithmic solutions to 3D visibility problems.

On the properties of lines tangent to geometric objects, we also published this year in the journal *Discrete and Computational Geometry* a proof of an important folklore theorem in computer graphics, which is that a polyhedron of complexity n that approximates a surface in some reasonable way has silhouettes of expected size $O(\sqrt{n})$, where the average is taken over all points of view [21].

The geometry of lines intersecting geometric objects also underlies many questions where lines play a prominent role, for instance in 3D visibility. Specifically, *isolated line transversals*, that is lines that intersect a collection of objects, but cannot be moved while maintaining an intersection with each object in the collection, are of particular importance as they capture transition phenomena. Over the last few years, we proved a “basis theorem” for isolated line transversals to disjoint balls: if ℓ is an isolated line transversal to a family F of disjoint balls in $\mathbb{R}^d$ then there exists a subfamily G of F of size at most $2d-1$ with the property that ℓ is an isolated line transversal to G. We proved that the constant $2d-1$ in this statement is best possible, in all dimensions [36]; as a result we obtained new lower bounds for the Helly number of families of transversals to disjoint balls, which in turn gives a lower bound on the combinatorial dimension of generalized linear programming formulations for finding a transversal to disjoint balls. We also extended our “basis theorem” to three-dimensional polyhedra [34] and to three-dimensional smooth convex objects.

Another problem in line geometry on which little is known is how the geometry of objects determines the structure of their line transversals. We proved that the set of directions of line transversals to disjoint balls in $\mathbb{R}^d$ in a given order is a convex subset of the space of directions. This has a number of applications in line geometry, for example tight combinatorial bounds and Helly-type theorems. These results have been presented at the 2007 SoCG conference [44] and appeared this year in the journal *Discrete and Computational Geometry* [13].

We also published a survey article on the properties of sets of line transversals to disjoint balls, to appear in a volume edited by the Institute for Mathematics and its Applications (IMA) dedicated to *Nonlinear Computational Geometry* [32], and a study of the problem of recognizing collections of balls obtained by slicing higher dimensional disjoint congruent balls by a linear space, to appear in the *International Journal of Computational Geometry* [12].

6.2. Exact geometric computing for low-degree surfaces

Participants: Jinsan Cheng, Laurent Dupont, Hazel Everett, Sylvain Lazard, Luis Penaranda, Maria Pentcheva, Sylvain Petitjean, Marc Pouget.

We developed, over the last years, the first efficient algorithm for computing an exact parameterization of the intersection of two quadric surfaces (ellipsoids, paraboloids, hyperboloids, etc.) in three-dimensional real space given by implicit equations with rational coefficients. We also produced a C++ implementation of this algorithm which is the only existing solution for computing exactly a parameterization of the intersection of two quadrics. A series of three articles describing the theoretical contributions of this work was published this year in the *Journal of Symbolic Computation* [17], [18], [19]

We have started working on the problem of computing the medial axis or Voronoi diagram of polyhedra in 3D. These structures are largely used in applications; the medial axis is, for instance, a way of representing a shape by its topological skeleton. Such a diagram is a partition of space into cells, each of which consists of the points closest to one particular object than to any other. Moreover, the set of points equidistant to two lines (or to a line and a point) is a quadric and the set of points equidistant to three lines is the intersection of two quadrics. While such structures are well-understood in the plane and for simple situations in higher dimensions (e.g. for sets of points), a lot remains to be done; for example, there is no working solution for computing exactly the medial axis of a polyhedron. We started by considering the Voronoi diagram of lines and we presented some very nice results characterizing the topology of the Voronoi diagrams of three lines. We proved, in particular, that the topology is invariant for lines in general position and we obtained a monotonicity property on the arcs of the diagram. We deduced a simple algorithm for sorting points along such an arc, which is presumably of great interest for future efficient algorithms for computing the medial axis of a polyhedron. The proof technique, which relies heavily upon modern tools of computer algebra, is also of great interest in its own right. These results were submitted to the journal *Discrete and Computational Geometry* in a special issue of selected papers from the 2007 Symposium of Computational Geometry (SoCG'07) [54]. We extended, this year, these results to the case of three lines in arbitrary positions providing the first complete characterization of the Voronoi diagram of any three lines [37]; these results also yielded a new algorithm, fundamental for handling robustness issues, for sorting points along the arcs of Voronoi diagrams of lines (with rational coefficients) using only rational linear tests.

We have presented a complete, exact and efficient algorithm and its implementation for computing the adjacency graph of an arrangement of integer quadrics [52]. This algorithm builds upon our previous work on parameterization of intersections of quadrics. Intersecting a parameterized intersection of two quadrics with a third quadric leads to find the real zeros of polynomials of degree at most 8 with possibly algebraic coefficients. Experiments show that our implementation outperforms past approaches when dealing with generic situations, even in case of large bitsize and/or algebraic coefficients. This efficiency, even over algebraic extensions and with large bit-size numbers, is due, during the computation of roots of univariate polynomials, to the use of the bitstream Descartes algorithm, which replaces each number by a series of certified approximations. In non-generic situations, the current implementation is hampered by slow gcd computations over algebraic extensions.

Since our work on quadrics can not easily be extended to more complex primitives (cubics, quartics, ...), an orthogonal research direction is to consider the (certified) approximation of complex shapes by lower-degree objects, with the idea that a simple, intermediate representation has great potential for dealing with curved objects. We previously worked in this direction by studying the approximation of smooth curves by tangent continuous conic splines, giving tight bounds on the complexity of the approximation by a conic spline with respect to the Hausdorff metric [56]. This year, we have started working on extensions of this result with a view to better grasp the complexity of the approximation, study the complexity of approximation by other types of primitives (conic biarcs for instance) and bound the time complexity of algorithms computing optimal approximants.

We also started working on the problem of geometric computing with algebraic tools. Our goal is two-fold. First, we want to make state-of-the-art algebraic software more accessible to the computational geometry community, in particular, through the computational geometric library CGAL. Second, our goal is to demonstrate to which extent such state-of-the-art certified algebraic root-finding systems can be used in geometric algorithms to obtain certified constructions involving curved objects without hindering performance. For that purpose, we focus on the problem of computing arrangements of algebraic curves in two dimensions. We have presented some results in these two directions at the 24th European Workshop on Computational Geometry (EuroCG'08) [28] and L. Penaranda should submit in 2008 a CGAL package which is a model of the *Univariate Algebraic Kernel* concept for algebraic computations (see section Software).

We worked over the last two years on the problem of computing (in a certified way) the topology of algebraic curves, that is, computing an arrangement of polylines isotopic to the input curve. The objective here is to compute efficiently and in a certified way arrangements of algebraic curves. A necessary key step is to compute the topology of any given curve. Moreover, geometric information, such as the position of singular and critical points, is also mandatory for computing arrangements of several curves using a sweep-line algorithm. A difficulty is to compute efficiently this information for the given coordinate system even if the curve is not in generic position; previous practical approaches shear back and forth the coordinate system, which is time consuming. In addition, costly computations with polynomials whose coefficients are algebraic should be avoided. We have recently presented an algorithm that incorporates several improvements over previous methods and overcome these difficulties [23]. In particular, our approach does not require generic position (nor shearing) and avoids the computations of sub-resultant sequences (in all cases). We have developed a Maple implementation of this algorithm which is very promising (see section Software). It is also worth mentioning that this topic of curve topology computation emerged from a collaboration with the SALSA (INRIA Rocquencourt) team while studying curves on surfaces related to differential geometric invariants [31].

We have started revisiting some key constant-complexity geometric problems with a view of better understanding their degenerate instances. The general idea is to find a more or less systematic approach for identifying low-degree predicates characterizing the fundamentally different instances of such a problem. For that, we rely on (classical) algebraic invariant century, which was perceived as a bridge between geometry and algebra by the mathematicians of the 19th century. In particular, we studied this year the relative position of two plane projective conics [33]. We showed that it can be characterized by predicates of bidegree at most $(6, 6)$ in the coefficients of the input conics, improving upon previous results. By relative position we mean the morphology of the intersection, the rigid isotopy class and which conic is inside the other when applicable. The predicates were derived by analyzing the algebraic invariant theory of pencils of conics and related constructions. We are working on extending these results to other primitives.

Finally, it should be noted that we also addressed other problems in computational geometry. We obtained, in particular, results on the complexity of Delaunay triangulation of random samples of a cylinder [25], in computational topology [29] and graph drawing [27], [22].

7. Other Grants and Activities

7.1. National initiatives

- *Lorraine Region*. The region of Lorraine supported our team in the context of a program for young teams and emerging project (3 Keuros this year).

7.2. International initiatives

7.2.1. Associated Teams and Other International Projects

- McGill-VEGAS associated team. This INRIA program is a joint project between our group and the computational geometry laboratory of McGill University (Montréal), and in particular Sue Whitesides. This associated team was started in 2002 under the name McGill-ISA before the creation of VEGAS. The research theme is 3D visibility [1], [2], [3], [4], [5], [6], [8], [20], [13], [15], [21], [24], [26], [30], [32], [46], [51], [55] and, more generally, computational geometry. In this context, we organize regular international workshops (1st to 7th McGill - INRIA Workshop on Computational Geometry in Computer Graphics, 2002 - 2008) which regroup, for one week, 15 to 25 researchers from around the world. Many research projects were initiated during these workshops on the theme of 3D visibility and line geometry [5], [2], [3] [20], [13], [24], [26], [22] [51], [46], [50], [42], [49]. Note finally that our Ph.D. student, L. Zhang, is co-supervised with S. Whitesides. Also, S. Whitesides has moved in August to the university of Victoria (Canada).

In the context of this cooperation, INRIA supported VEGAS up to 4 Keuros and McGill University supported the Canadian side up to 5 K$CAN in 2007.

- KAIST-INRIA associated team. This INRIA program is a joint project between VEGAS and the Theory of Computation Laboratory of the KAIST University of Daejeon, in Korea, more particularly the group of Otfried Cheong. The research theme is Discrete and Computational Geometry, in general, with a particular emphasis on questions where both continuous and discrete aspects come into play and interact. In 2008, this associated team supported mutual visits and a one-week workshop where members from both sides gathered to work on research questions.

  In 2008, this cooperation was supported for 23 kE by INRIA and for 4kE by our partners.

- Sylvain Petitjean started a collaboration with Pr. Gert Vegter of the University of Groningen on “Certified Geometric Approximation”. This collaboration is funded by the Netherlands Organization for Scientific Research (NWO) - 2008–2012. Alberto Llera started his PhD co-advised by S. Petitjean and G. Vegter this year.

7.3. Visiting scientists

International visitors:

- Andreas Holmsen, KAIST, Jul., 1 week;

- Otfried Cheong, KAIST, Nov., 1 week;

- Thorsten Theobald, Universität Frankfurt, Mar., 1 week.

Visitors from France:

- Cyril Nicaud, Univ. de Marne-la-Vallée, Jul., 1 week;

- Olivier Devillers, INRIA, Sophia, Jul., 1 week.

International visits:

- Xavier Goaoc, Kaist, S. Korea, Apr., 3 weeks + Jun., 1 week;

- Marc Pouget, Kaist, S. Korea, Sept., 1 week;

- Sylvain Lazard, McGill Univ. Canada, Aug. 1week ; Eindhoven Univ., The Netherlands, Aug. 1 week;

- Hazel Everett, McGill Univ. Canada, Aug. 1week ; Eindhoven Univ., The Netherlands, Aug. 1 week.

8. Dissemination

8.1. Teaching

All of the teaching activities were carried out in Nancy. The research Masters program is a joint degree with Univ. Nancy 1, Univ. Nancy 2 and the engineering school INPL. These three institutes are jointly known as University of Nancy.

Several members of the group, in particular the professors, assistant professors and Ph.D. students, actively teach at Université Nancy 2 and INPL. Other members of the group also teach in the Master of Computer Science of Nancy; namely H. Everett, S. Lazard offer the module “Computational geometry and robustness” and M. Pouget teaches a course on CGAL. X. Goaoc also intervenes in the Master’s program of the geology school at INPL with lectures on the same topic.

8.2. Visibility

Program and Paper Committees:

- Hazel Everett: Program committee of the Canadian Conference on Computational Geometry, (CCCG'08).
- Sylvain Petitjean: Program committee of the 24th ACM Symposium on Computational Geometry (SoCG'08). Paper committee of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR'08). Program committee of the ACM Solid and Physical Modeling Symposium (SPM'08). Program committee of the Shape Modeling International Conference (SMI'09). Paper committee of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR'09).
- All permanent team members: program committee of the European Workshop on Computational Geometry (EuroCG'08).

Workshop Organizations:

- Hazel Everett and Sylvain Lazard co-organized with S. Whitesides (McGill University) the 7th Workshop on Geometry Problems in Computer Graphics[1] (Bellairs Research Institute of McGill University) in Feb. (1 week).
- Xavier Goaoc co-organized a one-week workgroup (Ouessant) with Otfried Cheong (KAIST) in July (1 week).
- The Vegas team organized the European Workshop on Computational Geometry (EuroCG'08) in March (Nancy, 3 days) and co-organized the joint one-day workshop CGAL Innovations and Applications: Robust geometric software for complex shapes.

Thesis committees:

- Hazel Everett: external examiner of E. Moet, Uthrecht Univ, The Netherlands; committee member of C. Gray, Eindhoven Univ., The Netherlands.
- Sylvain Lazard: committee member of C. Wormser, Univ. of Nice Sophia-Antipolis, France.

Invited Talks:

- Xavier Goaoc, Oberwolfach workshop on discrete geometry, Sept., Germany.
- Sylvain Lazard, Annual Colloquium of Ecole Polytechnique (LIX), Emerging Trends in Visual Computing (ETVC'08), Nov, France. Journées Nationales de Calcul Formel, Oct., CIRM, France. Journées d'Approximation, Modélisation Géométrique et Applications de la SMAI-AFA Nov., CIRM, France.
- Sylvain Petitjean, Journées d'Approximation, Modélisation Géométrique et Applications de la SMAI-AFA, Nov., CIRM, France.
- Marc Pouget, Journées Nationales de Calcul Formel, Oct., CIRM, France.

Other responsibilities:

- Hazel Everett: Director of the Mathematics and Computer Science Department of Université Nancy 2. Vice-president of the hiring committee for computer science, Université Nancy 2. Member of the Research Council of Nancy-Université. Member of the "équipe de direction" of LORIA.
- Sylvain Lazard: Guest editor of *Comput. Geom.: Theory and App.* (special issue on selected papers from EuroCG'08). Guest editor (with L. Gonzalez-Vega, Santander Univ., Spain) of Mathematics in Computer Science (special issue on Computational Geometry and CAGD). Reviewer for the Natural Sciences and Engineering Research Council of Canada (NSERC).
- Laurent Dupont: Member of the hiring committee for computer science, Université Nancy 2.

---

[1] http://www.loria.fr/~everett/McGill-ISA/McGill-ISA.html

- Xavier Goaoc: Member of the hiring committee for computer science, Université Paris 6.
- Sylvain Petitjean: Scientific vice-delegate of INRIA Nancy Grand-Est. Member of the “équipes de direction” of LORIA and INRIA Nancy Grand-Est.
- Marc Pouget, Member of the CGAL Editorial Board.

8.3. Valorization and technology transfer

CIRTES. is a research company in rapid prototyping. We have exchanges with CIRTES since we started in 2000 co-supervising a CIFRE PhD student, G. Lauvaux [57].

The objective of fast prototyping in today’s manufacturing industry is to produce a physical model from a virtual one. Such a prototype should be cheap and easy to produce. Two popular technologies are NC machining in which an object is constructed using a 5 degrees of freedom drilling machine and layered manufacturing in which the object is built up from several layers.

Stratoconception® is the main layered-manufacturing technique developed at CIRTES. In this technique a polyhedron is manufactured in fairly thick slices (e.g. 1 cm), each slice being manufactured by a two and one-half degrees of freedom tool [43]. Not all objects can be perfectly manufactured that way, because some regions are not accessible to the drill, and choice of drilling direction and position of the slices have a large impact on the quality of the finished product. In fact, if the prototype is made of wood, manufacturing inaccuracies are repaired by manual sanding, a very time-consuming process. Once the positions of the slices have been chosen and the direction of drilling fixed, the error is exactly the same as the volume of the shadow cast on each slice by a light at infinity in the drilling direction.

Computing the optimal slicing strategy and drilling direction so as to minimize the volume of inaccessible regions is an open problem. We have proposed a practical approximate solution to this problem which has been tested with success and is now used by CIRTES [57]. Computing a provably optimal solution remains however an open problem. Note that the notion of 3D visibility is a key in this work since a tool can only drill material that is accessible, that is, that it can “see”.

9. Bibliography

Major publications by the team in recent years

[1] C. Borcea, X. Goaoc, S. Lazard, S. Petitjean. *Common Tangents to Spheres in* $\mathbb{R}^3$, in "Discrete and Computational Geometry", vol. 35, n^o 2, 2006, p. 287-300.

[2] H. Brönnimann, H. Everett, S. Lazard, F. Sottile, S. Whitesides. *Transversals to line segments in three-dimensional space*, in "Discrete and Computational Geometry", vol. 34, n^o 3, 2005, p. 381 - 390, http://hal.inria.fr/inria-00000384/en/.

[3] H. Brönnimann, O. Devillers, V. Dujmovic, H. Everett, M. Glisse, X. Goaoc, S. Lazard, H.-S. Na, S. Whitesides. *Lines and free line segments Tangent to Arbitrary Three-dimensional Convex Polyhedra*, in "SIAM Journal on Computing", vol. 37, 2007, p. 522-551, http://hal.inria.fr/inria-00103916/en/.

[4] S.-W. Cheng, O. Cheong, H. Everett, R. van Oostrum. *Hierarchical decompositions and circular ray shooting in simple polygons*, in "Discrete and Computational Geometry", vol. 32, n^o 3, 2004, p. 401–415.

[5] O. Cheong, X. Goaoc, A. Holmsen, S. Petitjean. *Hadwiger and Helly-type theorems for disjoint unit spheres*, in "Discrete & Computational Geometry", vol. 39, 2008, p. 194-212, http://hal.inria.fr/inria-00103856/en/.

[6] O. DEVILLERS, V. DUJMOVIC, H. EVERETT, X. GOAOC, S. LAZARD, H.-S. NA, S. PETITJEAN. *The expected number of 3D visibility events is linear*, in "SIAM Journal on Computing", vol. 32, n$^{\text{o}}$ 6, Jun 2003, p. 1586-1620.

[7] L. DUPONT, D. LAZARD, S. LAZARD, S. PETITJEAN. *Near-Optimal Parameterization of the Intersection of Quadrics: I. The Generic Algorithm; II. A Classification of Pencils; III. Parameterizing Singular Intersections*, in "Journal of Symbolic Computation", autres url : http://hal.inria.fr/inria-00186090/en/, http://hal.inria.fr/inria-00186091/en/, vol. 43, 2008, p. 168–191, 192–215, 216–232, http://hal.inria.fr/inria-00186089/en/.

[8] M. GLISSE, S. LAZARD. *An Upper Bound on the Average Size of Silhouettes*, in "Discrete & Computational Geometry", vol. 40, 2008, p. 241-257, http://hal.inria.fr/inria-00336571/en/.

[9] S. LAZARD, L. M. PEÑARANDA, S. PETITJEAN. *Intersecting Quadrics: An Efficient and Exact Implementation*, in "Computational Geometry, Theory and Applications", (Preliminary version in Proc. of 20th Annual Symposium on Computational Geometry (SoCG'04), New-York), vol. 35, n$^{\text{o}}$ 1-2, 2006, p. 74–99, http://hal.inria.fr/inria-00000380/en/.

[10] S. PETITJEAN. *A Survey of Methods for Recovering Quadrics in Triangle Meshes*, in "ACM Computing Surveys", vol. 34, n$^{\text{o}}$ 2, 2002, p. 211–262.

Year Publications

Doctoral Dissertations and Habilitation Theses

[11] J. DEMOUTH. *Événements Visuels de Convexes et Limites d'Ombres*, Thèse d'université, Université Nancy 2, November 2008.

Articles in International Peer-Reviewed Journal

[12] G. BATOG, X. GOAOC. *Inflating balls is NP-hard*, in "International Journal of Computational Geometry & Applications", 2008, http://hal.inria.fr/inria-00331423/en/.

[13] C. BORCEA, X. GOAOC, S. PETITJEAN. *Line transversals to disjoint balls*, in "Discrete and Computational Geometry", vol. 39, 2008, p. 158–173, http://hal.inria.fr/inria-00176198/en/.

[14] F. CAZALS, M. POUGET. *Jet fitting 3: A Generic C++ Package for Estimating the Differential Properties on Sampled Surfaces via Polynomial Fitting*, in "ACM Transactions on Mathematical Software", vol. 35, 2008, http://hal.inria.fr/inria-00329731/en/.

[15] O. CHEONG, X. GOAOC, A. HOLMSEN, S. PETITJEAN. *Hadwiger and Helly-type theorems for disjoint unit spheres*, in "Discrete & Computational Geometry", vol. 39, 2008, p. 194-212, http://hal.inria.fr/inria-00103856/en/.

[16] J. DEMOUTH, O. DEVILLERS, H. EVERETT, M. GLISSE, S. LAZARD, R. SEIDEL. *On the Complexity of Umbra and Penumbra*, in "Computational Geometry: Theory and Applications", To appear. (Special issue of selected papers from the 23rd European Conference on Computational Geometry, 2007.), 2008.

[17] L. DUPONT, D. LAZARD, S. LAZARD, S. PETITJEAN. *Near-Optimal Parameterization of the Intersection of Quadrics: I. The Generic Algorithm*, in "Journal of Symbolic Computation", vol. 43, 2008, p. 168–191, http://hal.inria.fr/inria-00186089/en/.

[18] L. DUPONT, D. LAZARD, S. LAZARD, S. PETITJEAN. *Near-Optimal Parameterization of the Intersection of Quadrics: II. A Classification of Pencils*, in "Journal of Symbolic Computation", vol. 43, 2008, p. 192–215, http://hal.inria.fr/inria-00186090/en/.

[19] L. DUPONT, D. LAZARD, S. LAZARD, S. PETITJEAN. *Near-Optimal Parameterization of the Intersection of Quadrics: III. Parameterizing Singular Intersections*, in "Journal of Symbolic Computation", vol. 43, 2008, p. 216–232, http://hal.inria.fr/inria-00186091/en/.

[20] H. EVERETT, S. LAZARD, B. LENHART, L. ZHANG. *On the Degree of Standard Geometric Predicates for Line Transversals in 3D*, in "Computational Geometry: Theory and Applications", To appear., 2008.

[21] M. GLISSE, S. LAZARD. *An Upper Bound on the Average Size of Silhouettes*, in "Discrete & Computational Geometry", vol. 40, 2008, p. 241-257, http://hal.inria.fr/inria-00336571/en/.

International Peer-Reviewed Conference/Proceedings

[22] D. BREMNER, J. LENCHNER, G. LIOTTA, C. PAUL, M. POUGET, S. STOLPNER, S. WISMATH. *A Note on alpha-Drawable k-Trees*, in "CCCG'08: Canadian Conference on Computational Geometry, Canada", 2008, http://hal-lirmm.ccsd.cnrs.fr/lirmm-00324589/en/.

[23] J. CHENG, S. LAZARD, L. M. PEÑARANDA, M. POUGET, F. ROUILLIER, E. P. TSIGARIDAS. *On The Topology of Planar Algebraic Curves*, in "Proc. of the 24th European Workshop on Computational Geometry, France Nancy", 2008, p. 213–216, http://hal.inria.fr/inria-00329754/en/.

[24] J. DEMOUTH, O. DEVILLERS, M. GLISSE, X. GOAOC. *Helly-type theorems for approximate covering*, in "Proceedings of the twenty-fourth annual symposium on Computational geometry - SCG '08, États-Unis d'Amérique Washington", ACM, 2008, p. 120–128, http://hal.inria.fr/inria-00331435/en/.

[25] O. DEVILLERS, J. ERICKSON, X. GOAOC. *Empty-ellipse graphs*, in "19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'08), États-Unis d'Amérique San Francisco", 2008, p. 1249–1256, http://hal.inria.fr/inria-00176204/en/.

[26] O. DEVILLERS, M. GLISSE, S. LAZARD. *Predicates for line transversals to lines and line segments in three-dimensional space*, in "24th Annual ACM Symposium Computational Geometry, États-Unis d'Amérique College Park, Maryland", M. TEILLAUD (editor), ACM, 2008, p. 174-181, http://hal.inria.fr/inria-00336256/en/.

[27] H. EVERETT, S. LAZARD, G. LIOTTA, S. WISMATH. *Universal Sets of n Points for 1-bend Drawings of Planar Graphs with n Vertices*, in "The 15th International Symposium on Graph Drawing Graph Drawing. 15th International Symposium, GD 2007, Sydney, Australia, September 24-26, 2007. Lecture Notes in Computer Science, Australie Sydney", vol. 4875, Springer-Verlag, 2008, p. 345-351, http://hal.inria.fr/inria-00189036/en/.

[28] S. LAZARD, L. MARIANO PEÑARANDA, E. P. TSIGARIDAS. *A CGAL-based Univariate Algebraic Kernel and Application to Arrangements*, in "European Workshop on Computational Geometry, France Nancy", 2008, p. 91–94, http://hal.inria.fr/inria-00336563/en/.

[29] E. WOLF CHAMBERS, E. COLIN DE VERDIÈRE, J. ERICKSON, S. LAZARD, F. LAZARUS, S. THITE. *Walking Your Dog in the Woods in Polynomial Time*, in "ACM Symposium on Computational Geometry, États-Unis d'Amérique College Park, Maryland", 2008, p. 101–109, http://hal.inria.fr/inria-00336497/en/.

[30] L. ZHANG, H. EVERETT, S. LAZARD, C. WEIBEL, S. WHITESIDES. *On the Size of the 3D Visibility Skeleton: Experimental Results*, in "Algorithms - ESA, Allemagne Karlsruhe", 2008, p. 805–816, http://hal.inria.fr/inria-00336502/en/.

Scientific Books (or Scientific Book chapters)

[31] F. CAZALS, J.-C. FAUGÈRE, M. POUGET, F. ROUILLIER. *Ridges and Umbilics of Polynomial Parametric Surfaces*, in "Geometric Modeling and Algebraic Geometry", B. JUTTLER, R. PIENE (editors), Springer, 2008, p. 141–159, http://hal.inria.fr/inria-00329762/en/.

[32] X. GOAOC. *Some Discrete Properties of the Space of Line Transversals to Disjoint Balls*, in "Non-linear Computational Geometry IMA Volume Series", I. EMIRIS, F. SOTTILE, T. THEOBALD (editors), Springer, 2008, http://hal.inria.fr/inria-00335946/en/.

[33] S. PETITJEAN. *Invariant-based characterization of the relative position of two projective conics*, in "Non-Linear Computational Geometry", I. EMIRIS, F. SOTTILE, T. THEOBALD (editors), Springer, 2008, http://hal.inria.fr/inria-00335968/en/.

Other Publications

[34] B. ARONOV, O. CHEONG, X. GOAOC, G. RÖTE. *Lines Pinning Lines*, manuscript, 2008.

[35] G. BATOG. *Événements Visuels de Polyèdres*, Master thesis, École Normale Supérieure, Cachan, 2008.

[36] O. CHEONG, X. GOAOC, A. HOLMSEN. *Lower Bounds for Pinning Lines by Balls*, manuscript, 2008.

[37] C. GILLOT. *Diagrammes de Voronoï de trois droites en position arbitraire*, Master's Thesis, Masters thesis, Nancy Université, June 2008.

References in notes

[38] *GMP: the GNU MP Bignum Library*, http://www.swox.com/gmp, The Free Software Foundation.

[39] *LiDIA: a C++ Library for Computational Number Theory*, http://www.informatik.tu-darmstadt.de/TI/LiDIA, Darmstadt University of Technology.

[40] *QI: a C++ package for parameterizing intersections of quadrics*, 2005, http://www.loria.fr/equipes/vegas/qi, LORIA, INRIA Lorraine, VEGAS project.

[41] L. ALONSO, F. CUNY, S. PETITJEAN, J.-C. PAUL, S. LAZARD, E. WIES. *The Virtual Mesh: A Geometric Abstraction for Efficiently Computing Radiosity*, in "ACM Transactions on Graphics", vol. 20, n^{o} 3, 2001, p. 169-201.

[42] H. ALT, M. GLISSE, X. GOAOC. *On the worst-case complexity of the silhouette of a polytope*, in "Proceedings of 15th Canadian Conference on Computational Geometry (CCCG'03)", Aug 2003, http://www.loria.fr/publications/2003/A03-R-081/A03-R-081.ps.

[43] C. BARLIER. *Le procédé de prototypage rapide par Stratoconception®*, in "Proc. European Forum on Rapid Prototyping, Palaiseau, France", 1992.

[44] C. BORCEA, X. GOAOC, S. PETITJEAN. *Line transversals to disjoint balls*, in "23rd Annual ACM Symposium on Computational Geometry 2007 - SoCG'07 Proceedings of the twenty-third annual symposium on Computational geometry, Corée, République de Gyeongju", ACM Press, 2007, p. 245-254, http://hal.inria.fr/inria-00176201/en/.

[45] H. BRÖNNIMANN, O. DEVILLERS, V. DUJMOVIC, H. EVERETT, M. GLISSE, X. GOAOC, S. LAZARD, H.-S. NA, S. WHITESIDES. *Lines and free line segments Tangent to Arbitrary Three-dimensional Convex Polyhedra*, in "SIAM Journal on Computing", vol. 37, 2007, p. 522-551, http://hal.inria.fr/inria-00103916/en/ .

[46] H. BRÖNNIMANN, O. DEVILLERS, S. LAZARD, F. SOTTILE. *Lines tangent to four triangles in three-dimensional space*, in "Discrete & Computational Geometry", vol. 37, 2007, p. 369-380, http://hal.inria.fr/inria-00000598/en/.

[47] F. CAZALS, M. POUGET. *Approximation of Ridges and Umbilics on Triangulated Surface Meshes*, in "CGAL User and Reference Manual", CGAL EDITORIAL BOARD (editor), CGAL Editorial Board, 2007, http://www.cgal.org/Manual/3.3/doc_html/cgal_manual/packages.html#Pkg:Ridges_3.

[48] F. CAZALS, M. POUGET. *Estimation of Local Differential Properties*, in "CGAL User and Reference Manual", CGAL EDITORIAL BOARD (editor), CGAL Editorial Board, 2007, http://www.cgal.org/Manual/3.3/doc_html/cgal_manual/packages.html#Pkg:Jet_fitting_3.

[49] O. CHEONG, X. GOAOC, H.-S. NA. *Disjoint Unit Spheres Admit At Most Two Line Transversals*, in "Proceedings of the 11th Annual European Symposium on Algorithms (ESA'03)", Sept. 2003, http://www.loria.fr/publications/2003/A03-R-071/A03-R-071.ps.

[50] O. CHEONG, X. GOAOC, H.-S. NA. *Geometric permutations of disjoint unit spheres*, in "Computational Geometry: Theory and Applications", vol. 30, 2005, p. 253-270.

[51] O. DEVILLERS, V. DUJMOVIC, H. EVERETT, S. HORNUS, S. WHITESIDES, S. WISMATH. *Maintaining Visibility Information of Planar Point Sets with a Moving Viewpoint*, in "International Journal of Computational Geometry & Applications", vol. 17, 2007, p. 297-304, http://hal.inria.fr/inria-00192927/en/.

[52] L. DUPONT, M. HEMMER, S. PETITJEAN, E. SCHOMER. *Complete, Exact and Efficient Implementation for Computing the Adjacency Graph of an Arrangement of Quadrics*, in "15th Annual European Symposium on Algorithms - ESA 2007 Algorithms ESA 2007 15th Annual European Symposium, Eilat, Israel, October 8-10, 2007. Proceedings Lecture Notes in Computer Science, Israël Eilat, Israel, October 8-10, 2007", vol. 4698, Springer Berlin / Heidelberg, Yossi Azar, Tel-Aviv U. and Microsoft Research uy Even, Tel-Aviv U. Amos

Fiat, Tel-Aviv U. (Chair) Seffi Naor, Technion and Microsoft Research, 2007, p. 633-644, http://hal.inria.fr/inria-00165663/en/.

[53] F. DURAND. *Visibilité tridimensionnelle : étude analytique et applications*, Ph. D. Thesis, Université Joseph Fourier - Grenoble I, 1999.

[54] H. EVERETT, D. LAZARD, S. LAZARD, E. D. M. SAFEY. *The Voronoi Diagram of Three Lines*, in "23rd Annual Symposium on Computational Geometry Proceedings of the twenty-third annual symposium on Computational geometry, Corée, République de Gyeongju", vol. Session 8A, ACM, Hee-Kap Ahn, Otfried Cheong, and Kyung-Yong Chwa, 2007, p. 255-264, http://hal.inria.fr/inria-00186085/en/.

[55] H. EVERETT, S. LAZARD, S. PETITJEAN, L. ZHANG. *On the Expected Size of the 2D Visibility Complex*, in "International Journal of Computational Geometry & Applications", vol. 17, 2007, p. 361-381, http://hal.inria.fr/inria-00103926/en/.

[56] S. GHOSH, S. PETITJEAN, G. VEGTER. *Approximation by conic splines*, in "Mathematics in Computer Science", 2007, http://hal.inria.fr/inria-00188456/en/.

[57] G. LAUVAUX. *La réalisation d'œuvres d'art par prototypage rapide avec le procédé de stratoconception*, Ph. D. Thesis, Université de Reims Champagne-Ardennes, France, LORIA, June 2005.

[58] J. LEVIN. *A parametric algorithm for drawing pictures of solid objects composed of quadric surfaces*, in "Communications of the ACM", vol. 19, n^{o} 10, 1976, p. 555-563.

[59] M. POCCHIOLA, G. VEGTER. *The visibility complex*, in "International Journal of Computational Geometry and Applications", vol. 6, n^{o} 3, 1996, p. 1-30.

[60] A. REQUICHA, H. VOELCKER. *Solid modeling: a historical summary and contemporary assessment*, in "IEEE Computer Graphics and Applications", vol. 2, n^{o} 1, 1982, p. 9-24.

[61] L. ZHANG, H. EVERETT, S. LAZARD, S. WHITESIDES. *Towards an Implementation of the 3D Visibility Skeleton*, in "23rd Annual ACM Symposium on Computational Geometry, Gyeongju Corée, République de", 2007, p. 131 - 132, http://hal.inria.fr/inria-00338095/en/.