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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Project-Team Concha

Complex Flow Simulation Codes based on High-order and Adaptive methods

Bordeaux - Sud-Ouest

Theme : Computational models and simulation

Activity Report

2010

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Concha is an INRIA Project-Team joint with University of Pau and Pays de l'Adour and CNRS (LMA, UMR 5142). It has been created as an 'équipe INRIA' in april 2007, and is an EPI since february 2008.

1. Team

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2. Overall Objectives

2.1. Objectives

The main objective of this project is the development of innovative algorithms and efficient software tools for the simulation of complex flow problems. Accurate predictions of physical quantities are of great interest in fluid mechanics, for example in order to analyze instabilities, predict forces acting on a body, estimate the flow through an orifice, or predict thermal conductivity coefficients. Due to the complex and highly nonlinear equations to be solved, it is difficult to know in advance how fine the spatial or temporal resolution should be and how detailed a given physical model has to be represented. We propose to develop a systematic approach to these questions based on auto-adaptive methods.

Most of the physical problems under consideration have a three-dimensional character and involve the coupling of models and extremely varying scales. This makes the development of fast numerical methods and efficient implementation a question of feasibility. Our contributions concern modern discretization methods (high-order and adaptivity) and goal-oriented simulation tools (prediction of physical quantities, numerical sensitivities, and inverse problems). Concrete applications originate from aerodynamics, viscoelastic flows, heat transfer, and porous media.

The goal of the **first phase** of the project is to develop flow solvers based on modern numerical methods such as high-order discretization in space and time and self-adaptive algorithms. Adaptivity based on a posteriori error estimators has become a new paradigm in scientific computing, first because of the necessity to give rigorous error bounds, and second because of the possible speed-up of simulation tools. A systematic approach to these questions requires an appropriate variational framework and the development of a posteriori error estimates and adaptive algorithms, as well as sufficiently general software tools able to realize these algorithms. To this end we develop a single common library written in C++ and study at hand of concrete applications the possible benefits and difficulties related to these algorithms in the context of fluid mechanics. The main ingredients of our numerical approach are adaptive finite element discretizations combined with multilevel solvers and hierarchical modeling. We develop different kinds of finite element methods, such as discontinuous (DGFEM) and stabilized finite element methods (SFEM), either based on continuous or non-conforming finite element spaces (NCFEM). The availability of such tools is also a prerequisite for testing advanced physical models, concerning for example turbulence, compressibility effects, and realistic models for viscoelastic flows.

The goal of the **second phase** is to tackle questions going beyond forward numerical simulations: parameter identification, design optimization, and questions related to the interaction between numerical simulations and physical experiments. It appears that many questions in the field of complex flow problems can neither be solved by experiments nor by simulations alone. In order to improve the experiment, the software has to be able to provide information beyond the results of simple simulation. Here, information on sensitivities with respect to selected measurements and parameters is required. The parameters could in practice be as different in nature as a diffusion coefficient and a velocity boundary condition. It is our long-term objective to develop the necessary computational framework and to contribute to the rational interaction between simulation and experiment.

The interdisciplinary collaboration is at the heart for this project. The composition of the project team consists of mathematicians and physicists, and we develop collaborations with computer scientists.

2.2. Highlights

- Successful validation of our code for viscoelastic flows by comparison with the literature, experimental data, and a commercial program.
- Optimality of an adaptive finite element method for the Stokes equations.

3. Scientific Foundations

3.1. Challenges related to numerical simulations of complex flows

First, we describe some typical difficulties in our fields of application which require the improvement of established and the development of new methods.

- Coupling of equations and models
  The general equations of fluid dynamics consist in a strongly coupled nonlinear system. Its mathematical nature depends on the precise model, but in general contains hyperbolic, parabolic, and elliptic parts. The spectrum of physical phenomena described by these equations is very large: convection, diffusion, waves... In addition, it is often necessary to couple different models in order to describe different parts of a mechanical system: chemistry, fluid-fluid-interaction, fluid-solid-interaction...
- Robustness with respect to physical parameters
  The values of physical parameters such as diffusion coefficients and constants describing different state equations and material laws lead to different behaviour characterized for example by the Reynolds, Mach, and Weissenberg numbers. Optimized numerical methods are available in many situations, but it remains a challenging problem in some fields of applications to develop robust discretizations and solution algorithms.
- Multiscale phenomena

The inherent nonlinearities lead to an interplay of a wide range of physical modes, well-known for example from the study of turbulent flows. Since the resolution of all modes is often unreachable, it is a challenging task to develop numerical methods, which are still able to reproduce the essential features of the physical phenomenon under study.

3.2. Stabilized and discontinuous finite element methods

The discontinuous Galerkin method [74], [72], [47], [46] has gained enormous success in CFD due to its flexibility, links with finite volume methods, and its local conservation properties. In particular, it seems to be the most widely used finite element method for the Euler equations [48]. On the other hand, the main drawback of this approach is the large number of unknowns as compared to standard finite element methods. The situation is even worse if one counts the population of the resulting system matrices. In order to find a more efficient approach, it seems therefore important to study the connections with other finite element methods.

In view of the ubiquous problem of large Péclet numbers, stabilization techniques have been introduced since a long time. They are either based on upwinding or additional terms in the discrete variational formulation. The drawback of the first technique is a loss in consistency which generally leads to large numerical diffusion. The grand-father of the second technique is the SUPG/GLS method [59], [73]. Recently, new approaches have been developed, which try do avoid coupling of the different equations due to the residuals. In this context we cite LPS (local projection stabilization) [66], [58], [51][1] and CIP (continuous interior penalty) [60], [61].

3.3. Finite element methods on quadrilateral and hexahedral meshes

The construction of finite element methods on quadrilateral, and particularly, hexahedral meshes can be a complicated task; especially the development of mixed and non-conforming methods is an active field of research. The difficulties arise not only from the fact that adequate degrees of freedom have to be found, but also from the non-constantness of the element Jacobians; an arbitrary hexahedron, which we define as the image of the unit cube under a tri-linear transformation, does in general not have plane faces, which implies for example, that the normal vector is not constant on a side.

In collaboration with Eric Dubach (Associate professor at LMAP) and Jean-Marie Thomas (Former professor at LMAP) we have built a new class of finite element functions (named pseudo-conforming) on quadrilateral and hexahedral meshes. The degrees of freedom are the same as those of classical iso-parametric finite elements but the basis functions are defined as polynomials on each element of the mesh. On general quadrilaterals and hexahedra, our method leads to a non-conforming method; in the particular case of parallelotopes, the new finite elements coincide with the classical ones [65], [64].

3.4. Finite element methods for interface problems

The NXFEM (Nitsche eXtended finite element method) has been developed in [68] and [69]. It is based on a pure variational formulation with standard finite element spaces, which are locally enriched in such a way that the accurate capturing of an interface not aligned with the underlying mesh is possible, giving a rigorous formulation of the very popular XFEM. A typical computation for the Stokes problem with varying, piecewise constant viscosity is shown in Figure 1. This technology opens the door to many applications in the field of fluid mechanics, such as immiscible flows, free surface flows and so on.

3.5. Adaptivity

Adaptive finite element methods are becoming a standard tool in numerical simulations, and their application in CFD is one of the main topics of Concha. Such methods are based on a posteriori error estimates of the discretization error avoiding explicit knowledge of properties of the solution, in contrast to a priori error estimates. The estimator is used in an adaptive loop by means of a local mesh refinement algorithm. The mathematical theory of these algorithms has for a long time been bounded to the proof of upper and lower bounds, but has made important improvements in recent years. For illustration, a typical sequence of adaptively refined meshes on an L-shaped domain is shown in Figure 2.

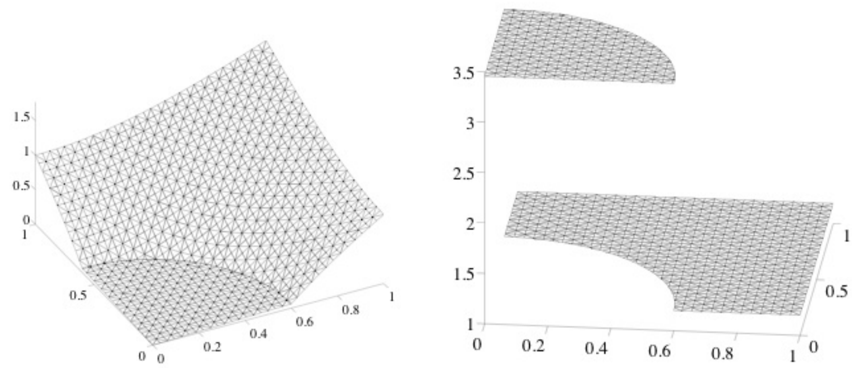


Figure 1. Incompressible elasticity with discontinuous material properties (left: modulus of velocities, right: pressure; from [49]).

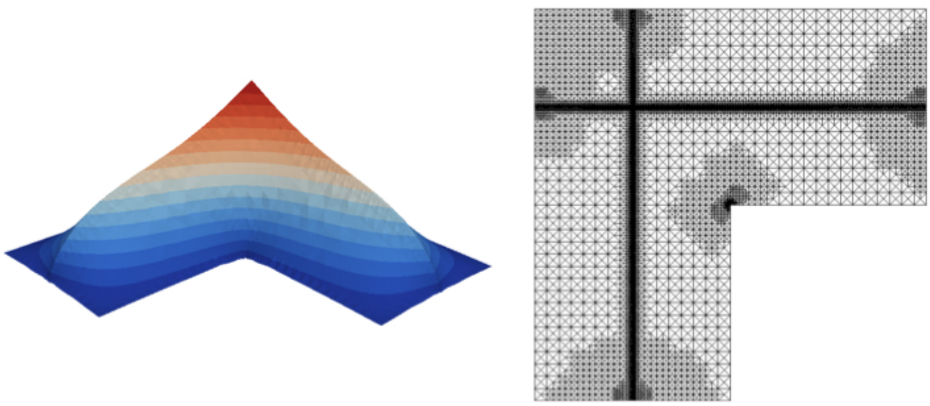


Figure 2. Solution with rough right-hand-side in a corner domain and adaptively refined mesh (from [53]).

The theoretical analysis of mesh-adaptive methods, even in the most standard case of the Poisson problem, is in its infancy. The first important results in this direction concern the convergence of the sequence of solution generated by the algorithm (the standard a priori error analysis does not apply since the global mesh-size does not necessarily go to zero). In order to prove convergence, an unavoidable data approximation term has to be treated in addition to the error estimator [76]. These result do not say anything about the convergence speed, that is the number of unknowns required to achieve a given accuracy. Such complexity estimates are the subject of active research, the first fundamental result in this direction is [56].

Our first contribution [5] to this field has been the introduction of a new adaptive algorithm which makes use of an adaptive marking strategy, which refines according to the data oscillations only if they are by a certain factor larger then the estimator. This algorithm allowed us to prove geometric convergence and quasi-optimal complexity, avoiding additional iteration as used before [82]. We have extended our results to conforming FE without inner node refinement [54] and to mixed FE [53]. In this case, a major additional difficulty arises from the fact that, due to the saddle-point formulation, the orthogonality relation known from continuous FEM does not hold. In addition, we have considered the case of incomplete solution of the discrete systems. To this end, we have developed a simple adaptive stopping criterion based on comparison of the iteration error with the discretization error estimator, see also [52].

Goal-oriented error estimation has been introduced in [55]. It allows to error control and adaptivity directly oriented to the computation of physical quantities, such as the drag and lift coefficient, the Nusselt number, and other physical quantities.

4. Application Domains

4.1. Aerodynamics

Aerodynamics provide a challenging field for numerical simulations in fluid dynamics with a wide range of applications. Robustness of the simulation software with respect to physical parameters as the Reynolds and Mach numbers is necessary condition. In general, realistic simulations need to be done in three dimensions, which makes the efficiency of the numerical approach and implementation a question of feasibility. Therefore, different efforts are made in this project in order to tackle these subjects.

4.2. Viscoelastic flows

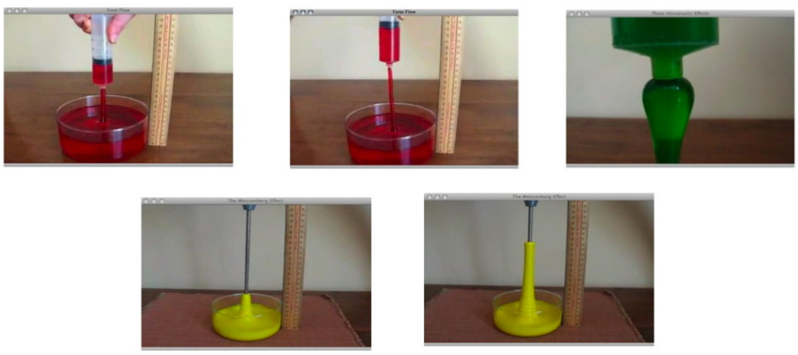


Figure 3. Unexpected behavior of flow of polymer liquids.

Polymeric fluids are, from a rheological point of view, viscoelastic non-Newtonian fluids, see Figure 3. Their specific behavior can be observed in a variety of physical phenomena, which are unseen with Newtonian liquids and which cannot be predicted by the Navier-Stokes equations. The better known examples include the rod climbing Weissenberg effect, die swell and extrusion instabilities (cf. fig. 1). The rheological behavior of polymers is so complex that many different constitutive equations have been proposed in the literature in order to describe these phenomena, see for instance [77]. The choice of an appropriate constitutive law is still a central problem. We consider realistic constitutive equations such as the Giesekus model. In comparison to the classical models used in CFD, such as UCM or Oldroyd B fluids, the Giesekus model is characterized by a quadratic stress term. It is important to understand the theoretical properties of the Giesekus model. As outlined above, energy estimates are crucial for the development of robust numerical schemes, see also the recent work on similar questions in the EPI MICMAC [57], [70].

Our aim is to develop new algorithms for the discretization of polymer models, which should be efficient and robust for $We > 10$. For this purpose, we will develop a mathematical approach based on recent ideas on discretizations preserving the positivity of the conformation tensor. This property is believed to be crucial in order to avoid numerical instabilities associated with large Weissenberg numbers. In order to develop monotone numerical schemes, we use recent discretization techniques such as stabilized finite element and discontinuous Galerkin methods. We have validated our code at hand of academic benchmark problems in comparison with the commercial code PolyFlow®.

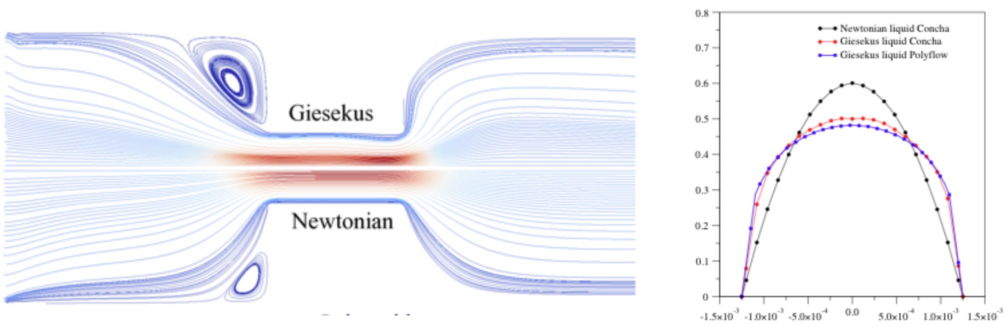


Figure 4. Comparison of Newtonian and Giesekus flow for a contraction flow (left) and comparison with PolyFlow® (right).

The result of a computation of a 4:1-contraction, comparing Newtonian flow with Giesekus model, is shown in Figure 4. In the same figure, a comparison of the computed profile in the channel with the one obtained by the PolyFlow®, both on a relatively coarse mesh, is shown. A precise study shows that the results are in good agreement for moderate Weissenberg numbers We; the computation time is by a factor of two smaller for the preliminary version of our code based on triangular meshes. For $We > 20$, we were not able to get a converged solution with the commercial code, whereas our program yields stationary solutions up to $We \approx 30$.

Further improvements are expected from the use of adaptivity, as well as from the implementation of adequate iterative solvers. The long-term goal is to successively build up robust and efficient software tools in order to tackle design problems, such as the design of mixing devices.

4.3. Heat transfer

Heat transfer problems involve the coupling of the flow field of the fluid with temperature inside the flow and possibly on the boundary of the flow domain. A typical example of a heat transfer problem is the cooling of a combustion engine, see the project Optimal described in Section 7.1.

4.4. Turbulence

Turbulent flows are ubiquitous in industrial applications. Direct numerical simulation (DNS), which aims at complete resolution of the flow field up to the Kolmogorov scale, has historically been limited to very simple geometries. The increase of computational power and the development of specialized numerical methods open the door to a wider range of applications. However, for most applications of practical interest, the use of some kind of turbulence modeling is un avoidable in order to obtain the prediction of averaged values. In this field, we are particularly interested in variational multiscale methods and its relations to stabilized finite element methods.

5. Software

5.1. C++ library Concha

Participants: Roland Becker, Daniela Capatina, Robert Luce, David Trujillo.

The objectives of our library CONCHA are to offer a flexible and extensible software with respect to:

- Numerical methods and
- Physical models.

The aim is to have a flexible code which could easily switch between the different discretizations, in order to provide a toolbox for rapid testing of new ideas.

The software architecture is designed in such a way that a group of core developers can contribute in an efficient manner, and that independent development of different physical applications is possible. Further, in order to accelerate the integration of new members and in order to provide a basis for our educational purposes (see Section 9.3), the software proposes different entrance levels. The basic structure consists of a common block, and several special libraries which correspond to the different fields of applications described in Sections 4.1–4.4 Hyperbolic solvers, Low-Mach number flow solvers, DNS, and viscoelastic flows. A more detailed description of each special library may be found below. In order to coordinate the cooperative development of the library, Concha is based on the INRIA-Gforge.

5.2. User interface and python interface

Participants: Guillaume Baty, David Trujillo.

A graphical user-interface facilitate the use of the C++-library. It has been developed by Guillaume Baty in collaboration with Pierre Puiseux (associate professor at LMAP). All members of the team have been involved in the testing of the interface. The first objective is to provide an easy way of installation and to facilitate the usage. To this end we use the python language with Qt in order to take advantage of higher level libraries, which allow us to reduce development time.

We are confronted with heterogenous backgrounds and levels of implication of the developers and users. It seems therefore crucial to be able to respond to the different needs. Our aim is to facilitate the development of the library, and at the same time, to make it possible that our colleagues involved in physical modeling can have access to the functionality of the software with a reasonable investment of time. Two graphical user interfaces have been developed: one for the installation of the library and another one for the building and execution of projects. They are based on common database and scripts written in python. The scripts can also be launched in a shell. In Figure 5 the user interface of the install tool is shown. The option panel allows to choose the components for conditional compilation and the compilation type (debug and release).

In Figure 6 the user interface of the project tool is shown. A project consists of a number of sources files and a parameter file used by the C++-executable. The sources define classes derived from the library, which are used to specify certain data such as boundary conditions and employed finite element spaces. The parameter file contains algorithmic information and physical parameters. It is generated from a database by the python utilities.

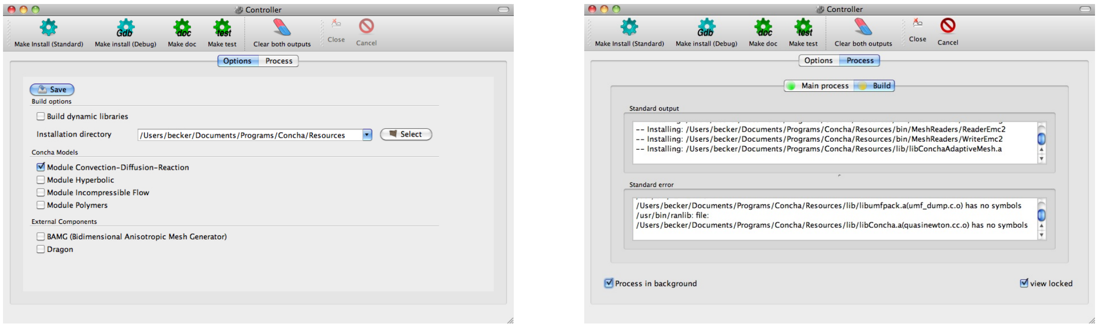


Figure 5. Graphical user interface: option panel (left) and process panel (right) of the install tool.

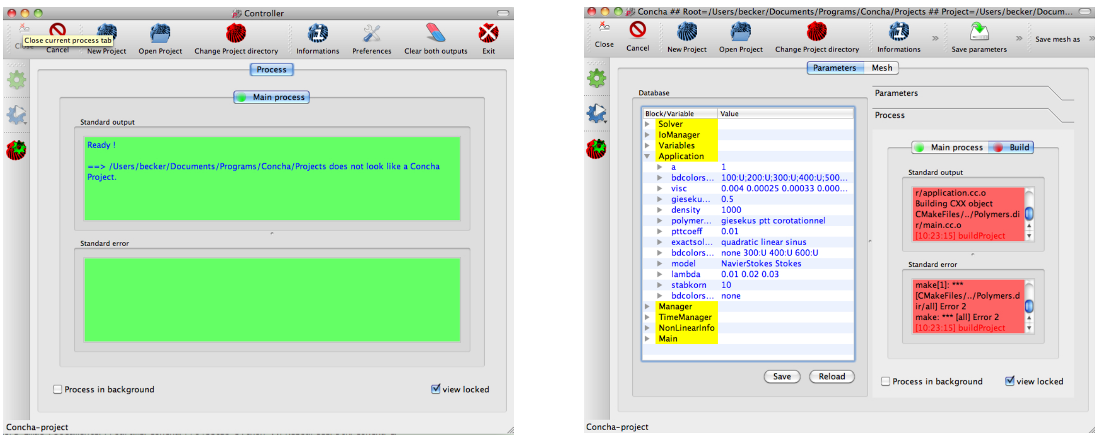


Figure 6. Graphical user interface: project build panel (left) and parameter panel (right) of the project tool.

The tools offered by this development platform are based on a python interface for the library, called pyConcha. It offers a common interface, based on a pluggin-system, which allows the devloppement of command line tools in parallel. This year the consolidation of the interface part of pyConcha has been an important task. The pyConcha library is now a framework rather than a simple interface to Concha C++ library. It allows now creation of plugins, so that each user-programmer can customize pyConcha to his own goals. Previously, two main programs where working: concha-install.py to install library, and concha-project.py for (semi-)end-users. Both are now plugins of pyConcha, and can be launched by pyConcha at startup. A plugin visualization could now be developped in an independant way, and launched by pyConcha on demand.

The structure of pyConcha framework is clearly splitted in various modules(layers) : Command Line Interface module, Graphical User Interface module and Handlers modules, see Figure 7. A great effort has been made for internationalization of pyConcha.

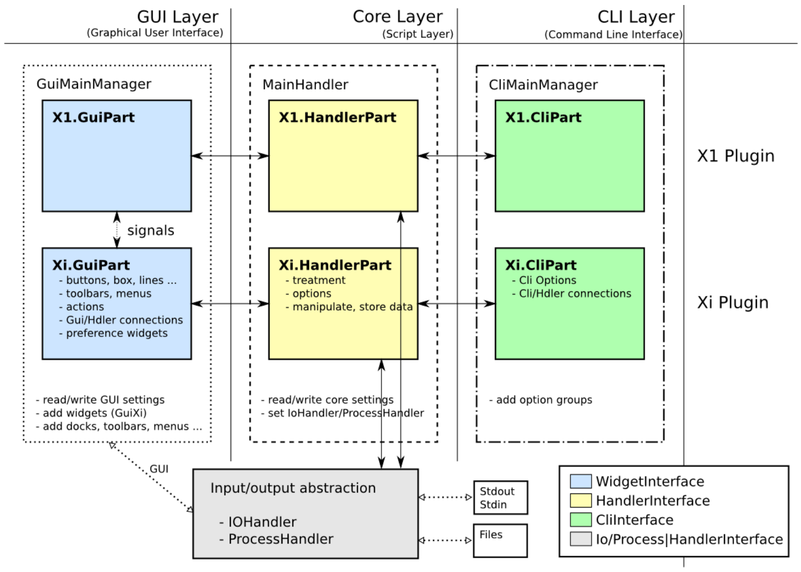


Figure 7. Structure of the pyConcha framework.

5.3. Parallelization

Participants: Roland Becker, David Trujillo, Elies Bergounioux.

We have started this year the parallelization of the library in collaboration with the INRIA-team Runtime. Marie-Christine Counilh and Olivier Aumage (Runtime) have analyzed the structure of the library in order to develop a strategy for the parallelization based on OpenMP. In parallel, Elies Bergounioux has started to develop a version of the library using MPI. The idea behind this development is to provide an implementation of parallel algorithms independent of a specific tools for communication (MPI, sequential parallelization).

5.4. Euler equations

Participants: Roland Becker, Robert Luce, Eric Schall.

Based on the library CONCHA we have develop a solver for hyperbolic PDE's based on DGFEM. So far different standard solvers for the Euler equations such as Lax-Friedrichs, Steger-Worming, and HLL have been implemented for test problems. A typical example is the scram jet test case shown in Figure 8.

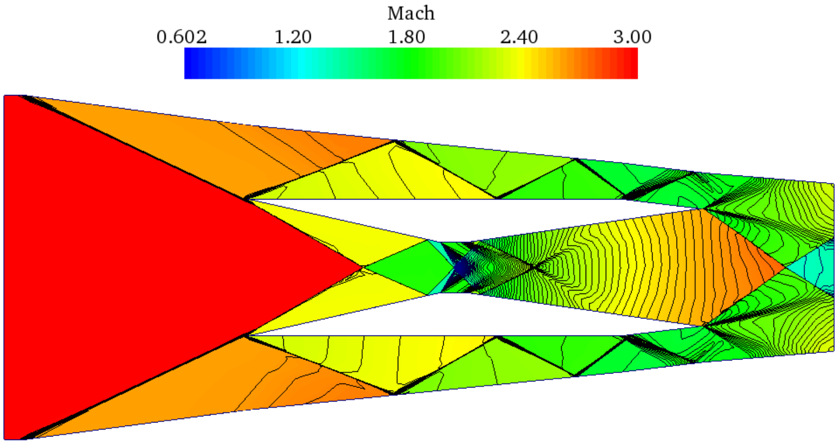


Figure 8. Computed Mach-number distribution for the Scramjet test problem.

5.5. Incompressible flow solvers

Participants: Roland Becker, Daniela Capatina, Robert Luce, David Trujillo.

We have started the validation of the implementation of different finite element methods for incompressible flows at hand of standard benchmark problems as the Stokes flow around a symmetric cylinder [71] and the stationary flow around a slightly non symmetric cylinder [81], see Figure 9.

5.6. DNS

Participants: Roland Becker, David Trujillo, Elies Bergounioux.

For the direct numerical simulation of incompressible turbulent flows, we have started to develop a special solver based on structured meshes with a fast multigrid algorithm incorporating projection-like schemes. The main idea is to use non-conforming finite elements for the velocities with piecewise constant pressures, leading to a special structure of the discrete Schur complement, when an explicit treatment of the convection and diffusion term is used.

5.7. Polymer flow

Participants: Roland Becker, Daniela Capatina, Julie Joie, Didier Graebling.

Based on our library CONCHA we have implemented a three-field formulation with unknowns (u, p, τ) for the two-dimensional Navier-Stokes equations, based on nonconforming finite elements. The extension to the Giesekus-model for polymers has been achieved, see Section 6.4. In the case of Newtonian flows, the extra-tensor can be eliminated in order to reduce storage and computing time. This procedure serves as a pre-conditioner in the general case. The aim is to provide software tools for the problems in Section Viscoelastic flows.

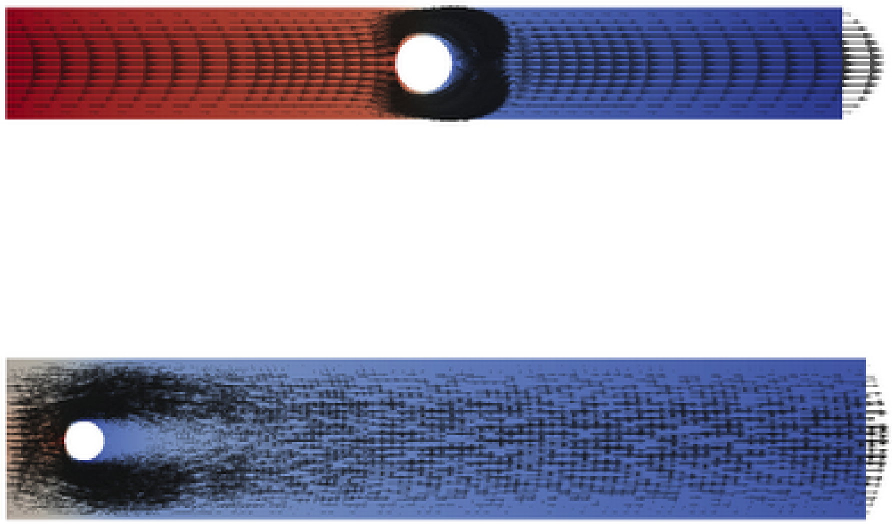


Figure 9. Flow fields for the Stokes (above) and Navier-Stokes (below) benchmark.

5.8. Validation and comparison with other CFD-software

Participants: Daniela Capatina, Didier Graebling, Julie Joie, Eric Schall.

We intend to compare computations based on CONCHA with other codes at hand of the prototypical test problems described above. This allows us to evaluate the potential of our numerical schemes concerning accuracy, computing time and other practical expects such as integration with mesh generators and post-processing. At the same time, this, unfortunately very time-consuming, benchmarking activity allows us to validate our own library. The following commercial and research tools might be considered: *Aéro3d (INRIA-Smash), AVBP (CERFACS), ELSA (ONERA), Fluent (ANSYS), and OpenFOAM (OpenCfd), and Polyflow® (ANSYS)*. So far, we have compared our code for the Giesekus model of polymer flows with the commercial software Polyflow®, see Section 4.2.

6. New Results

6.1. Convergence of adaptive finite element algorithms

Participants: Roland Becker, Shipeng Mao, David Trujillo.

The theoretical analysis of mesh-adaptive methods is a very active field of research. In this year, we have generalized our previous results concerning optimality of adaptive methods to nonconforming finite elements [21]. Our results include the error due to iterative solution of the system matrices by means of a simple stopping criterion related to the error estimator. The main difficulty was the treatment of the nonconformity which leads to a perturbation of the orthogonality relation at the heart of the proofs for conforming finite elements. We have been able to extend this result to the Stokes equations, considering different lowest-order nonconforming finite elements on triangular and quadrilateral meshes [17].

In [18] we have shown that the smallness assumption required in all former proofs of optimality of adaptive finite element methods can be overcome in at least some situations.

Our theoretical studies, which are motivated by the aim to develop better adaptive algorithms, have been accompanied by software implementation with the Concha library, see Section 5.1. It hopefully opens the door to further theoretical and experimental studies.

6.2. Finite element methods for interface problems

Participants: Roland Becker, Robert Luce.

The original formulation of NXFEM [68] is based on the doubling of elements. In some situations, as the case of a moving interface, it is computationally more convenient to have a method with local enrichment, as for the standard XFEM. In [13] we have developed such an approach based on NXFEM.

One of the technical difficulties is the simultaneous robustness of the method with respect to the size of the intersection of a mesh cell with the interface and with respect to the discontinuous diffusion parameters.

6.3. Discontinous finite element methods

Participants: Roland Becker, Daniela Capatina, Julie Joie, Nour El Houda Seloula.

We have developed a new discontinuous Galerkin scheme for the Stokes equations and corresponding three-field equations. In this work, which is part of the Phd Thesis of Julie Joie, we introduce a modification of the stabilization term in the standard DG-IP method. This allows for a cheaper implementation and has a more robust behavior with respect to the stabilization parameter; we have shown convergence towards the solution of non-conforming finite element methods for linear, quadratic and cubic polynomial degrees. This scheme has been extended to the three-field formulation of the Stokes problem, which is a further step towards the polymer project of Section 4.2. Since it is well known that the non-conforming finite element approximations do not verify the discrete Korn inequality, an appropriate further stabilization term is introduced, see [50].

6.4. Viscoelastic fluids modeling and numerical simulation

Participants: Roland Becker, Daniela Capatina, Didier Graebling, Julie Joie.

We have continued our activities with respect to numerical simulations of polymer flows.

We have used the mixed non-conforming/DG method for the Giesekus mode (JACA), previously implemented on triangular meshes. We have further validated the Giesekus code by means of comparisons with experimental data [78] (cf Figure 11) in the 4:1 contraction and with semi-analytical solutions for Poiseuille flow [75], [79].

In view of the extension to 3D, we have developed a numerical scheme on quadrilateral meshes, based on the Rannacher-Turek non-conforming finite elements. The analysis has outlined the necessity of adding a new regularization term, in order to retrieve optimal error estimates for the underlying Stokes problem. Moreover, our formulation is close to well-known four-fields methods especially designed for non-Newtonian flows, such as EVSS or DEVSS [80], [67].

This scheme has been tested in particular on the benchmark case of flow around a 2D cylinder, for which our code converged for high values of the Weissenberg number (We > 70), see Figure 12. We have computed the drag and compared it with numerical data found in the literature [63], [71], [83], [62] in the case of the Oldroyd-B model. Our results are in good agreement with these data, see Figure 13.

We could solve the system on rather fine meshes (of about 10^6 elements) thanks to a multigrid solver based on a Vanka type preconditionner.

The major part of these results can be found in the PhD thesis of Julie Joie and was presented at several conferences.

Further developments of adequate iterative solvers such as the multigrid methods, as well as the use of adaptivity (eccomas) are expected to improve the simulations. With these tools, we envisage to simulate more realistic flows taking into account 3D geometries and thermal effects.

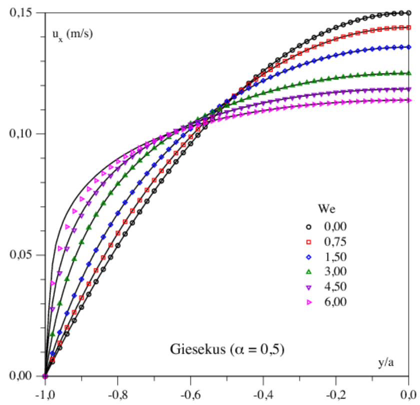
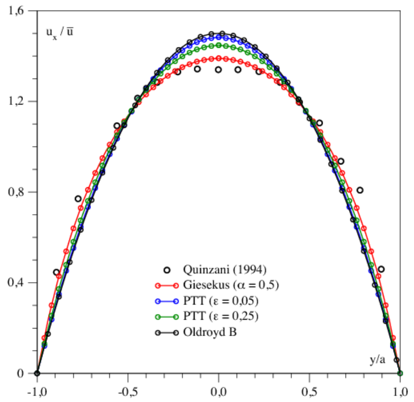


Figure 10. Comparison of velocity profile for different Weissenberg numbers.



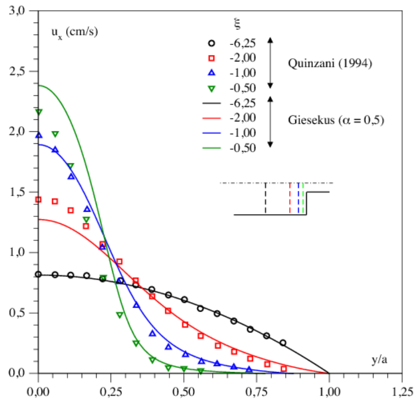


Figure 11. Comparison with experimental data for the 4:1 contraction.

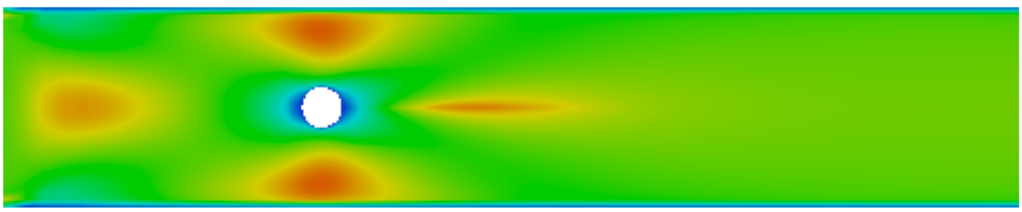


Figure 12. Flow around cylinder at $We = 75$.

λ	CONCHA	Dou *et al.*	Hulsen *et al.*	Étienne *et al.*	Damanik *et al.*
0	132,357	131,809	132,358	132,33	-
0,3	123,190	123,514	123,193	123,41	123,194
0,6	117,780	120,485	117,792	-	117,779

Figure 13. Comparison of drag coefficients with the literature.

The long-term goal is to successively build up robust and efficient software tools in order to tackle design problems. Moreover, we intend to consider other models of non-Newtonian fluids employed in other application domains such as biomedecine or agro-alimentary industries.

6.5. Finite element methods on quadrilateral and hexahedral meshes

Participant: Robert Luce.

Our approach to develop pseudo-conforming Lagrange finite element has been extended to mixed finite elements [45] on general quadrilaterals and hexahedra with plane faces.
The loss of convergence in $H(div)$-norm when we use RT, BDM or BDFM finite elements on non-parallelepipedic meshes is now well known and the new finite finite elements obtained allow us to obtain the optimal order of convergence.
The study and the use of non-conforming finite element of Lagrange type are very popular but lightly developed for finite elements of mixed type. So one of the originality of this work lies in the study of non-conforming **mixed** finite elements.
The main restriction in this study concerns the planarity of the faces, because of the non-constant normals on the faces it is difficult to adapt our approach to this type of element. But, more generally, the consequences of the use of hexahedra with no plane faces in many finite element or finite volume methods are not completely clear and lots of problems remain open.

7. Contracts and Grants with Industry

7.1. Optimal

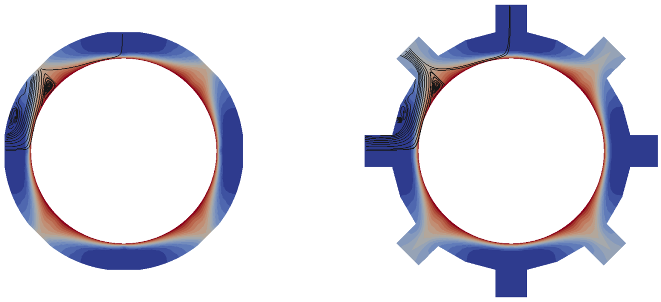
Participants: Roland Becker, Kossivi Gokpy, Eric Schall, David Trujillo.

Optimal is a research project related to the cooling of the stator of a turbomachinery. Both physical experiments and numerical simulations are employed. This project has three industrial (Liebherr, Epsilon, and SIBI) and three academic partners (Universities of Pau, Poitiers, and Toulouse). It has been evaluated by the cluster Aerospace Valley. The PhD-thesis of Kossivi Gokpy is financed by this project.

Our contributions concern the numerical simulation of the viscous flow in different geometrical configurations. Comparison with experimental data will be investigated with respect to the Nusselt number. The computed temperature and streamlines for typical geometries are shown in Figure 14. In addition, the computed Nusselt numbers for the two configurations and varying inflow velocities are given.

Among the different questions concerning modeling such as the boundary conditions at the in- and outlets and the sensitivity to the geometry, a particular point of interest is the study of compressibility effects.

The experimental part of the product is conducted in collaboration with Mathieu Mory, professor at UPPA, and the post-doctoral position of Stéphane Soubacq, who started to work in 10/2009, is financed by the project. The modeling and numerical simulation is done in collaboration with Abdellah Saboni, professor at UPPA.



v_{in}	geo1	geo2
10	135,97	130,53
20	216,73	213,59
50	352,61	341,47

Figure 14. Temperature fields and recirculation for two geometries (left and middle) and computed Nusselt numbers for different inflow velocities.

8. Other Grants and Activities

8.1. Regional Initiatives

8.1.1. Polymer simulation (Conseil Régional Aquitaine)

Participants: Roland Becker, Daniela Capatina, Didier Graebling, Julie Joie.

The objective of this project is the development of a robust simulation tool for polymer flows. To this end it seems unavoidable to investigate the reasons for the high-Weissenberg number problem, and consequently derive a robust approach.

The Phd-fellowship of Julie Joie is financed by this project. The objective of this work is to initiate the development of robust solvers for polymer liquids and it contributes to Section 5.7.

8.2. European Initiatives

We have long-standing the following international collaborations:

- Chalmers University of Technology, Sweden (Prof. Peter Hansbo)
  This collaboration concerns the following subjects: stabilized finite element methods, NXFEM, adaptivity.
- University of Sussex, UK (Prof. Erik Burman)
  This collaboration concerns the following subjects: stabilized finite element methods, NXFEM, adaptivity.
- University of Kiel, Germany (Prof. Malte Braack)
  This collaboration concerns the following subjects: stabilized finite element methods, fluid-acoustic interaction.

8.3. International Initiatives

We have a collaboration with the Institute of Computational Mathematics of the Chinese Academy of Sciences CAS (Prof. Lin Qun, Prof Li Yuan, Dr. Shipeng Mao, Dr. Mingxia Li) on finite element methods and numerical fluid mechanics. The team leader has been invited by Prof. Yuan for a one-month during summer at CAS in Bejing. It is planned that a post-doc of CAS will work for one year in Pau.

9. Dissemination

9.1. Animation of the scientific community

We have organized the sixth VMS (Variational MultiScale methods) workshop https://sites.google.com/site/conchapau/vms, which has allowed the scientific exchange of about 40 experts in the field of numerical methods for multiscale problems mostly related to CFD.

9.2. Phd-thesis

Two Phd-thesis have been defended this year. In november 2010, Julie Joie has defended her work on the numerical simulation of polymer flows [10]; Nour El Houda Seloula [11] passed her 'soutenance' in december 2010 concerning her work on non-standard boundary conditions for the Stokes and Navier-Stokes equations.

9.3. Teaching

The LMA has proposed a new Master program starting in 2007, which is called MMS (Mathématiques, Modélisation et Simulation) and has a focus on analysis, modeling, and numerical computations in PDEs; Robert Luce and R. Becker are co-responsables of this Master program. The core of this education is formed by lectures in four fields : PDE-theory, mechanics, numerical analysis, and simulation tools.

This master program includes lectures on physical applications, one of the three proposed application fields is CFD; lectures are provided by the members of the project; especially the following lectures have been given:

- Simulation numérique 1, Robert Luce and Eric Dubach,
- Analyse numérique des EDP, R. Becker and D. Capatina,
- Simulation numérique 2, Robert Luce and Eric Dubach,
- Méthodes numériques pour les EDP, R. Becker,
- Mécanique des fluides, R. Becker,
- Simulation numérique 3, P. Puiseux
- Mécanique des Fluides et Turbulence, Eric Schall, D. Graebling

The second semester of the second year is devoted to internships either in industry (which defines a practical means of collaboration with our industrial partners such as CERFACS, ONERA, TOTAL, and Turbomeca) or in research laboratories. In springtime 2010, Daniela Capatina has supervised the internship of Nelly Barrau entitled "Numerical simulation of anisothermal flows of Newtonian fluids" and the first-year project of F. Lassagne "Approximation of the temperature of a newtonien fluid with thermo-dependent viscosity".

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