IN PARTNERSHIP WITH:
CNRS
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Activity Report 2011

Team GECO

Geometric Control Design

RESEARCH CENTER
Saclay - Île-de-France

THEME
Modeling, Optimization, and Control of Dynamic Systems

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Team GECO

Keywords: Automatic Control, Nonlinear Control, Quantum Chemistry, System Analysis And Control, Tracking

Hosted at CMAP, École Polytechnique.

1. Members

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Moussa Gaye [since 10/2011, Fondation Hadamard scholarship]

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2. Overall Objectives

2.1. Highlights

- Starting from May 2011 the team has been created!
- Guilherme Afonso Mazanti, student at École Polytechnique, won the *Grand prix de stage de recherche* by École Polytechnique for his stage made in 2011 under the supervision of Yacine Chitour and Mario Sigalotti.

3. Scientific Foundations

3.1. Geometric control theory

The main research topic of the project-team will be **geometric control**, with a special focus on **control design**. The application areas that we target are control of quantum mechanical systems, neurogeometry and switched systems.

Geometric control theory provides a viewpoint and several tools, issued in particular from differential geometry, to tackle typical questions arising in the control framework: controllability, observability, stabilization, optimal control... [17], [53] The geometric control approach is particularly well suited for systems involving nonlinear and nonholonomic phenomena. We recall that nonholonomicity refers to the property of a velocity constraint that is not equivalent to a state constraint.

The expression **control design** refers here to all phases of the construction of a control law, in a mainly open-loop perspective: modeling, controllability analysis, output tracking, motion planning, simultaneous control algorithms, tracking algorithms, performance comparisons for control and tracking algorithms, simulation and implementation.

We recall that

- **controllability** denotes the property of a system for which any two states can be connected by a trajectory corresponding to an admissible control law ;
- **output tracking** refers to a control strategy aiming at keeping the value of some functions of the state arbitrarily close to a prescribed time-dependent profile. A typical example is **configuration tracking** for a mechanical system, in which the controls act as forces and one prescribes the position variables along the trajectory, while the evolution of the momenta is free. One can think for instance at the lateral movement of a car-like vehicle: even if such a movement is unfeasible, it can be tracked with arbitrary precision by applying a suitable control strategy;
- **motion planning** is the expression usually denoting the algorithmic strategy for selecting one control law steering the system from a given initial state to an attainable final one;
- **simultaneous control** concerns algorithms that aim at driving the system from two different initial conditions, with the same control law and over the same time interval, towards two given final states (one can think, for instance, at some control action on a fluid whose goal is to steer simultaneously two floating bodies.) Clearly, the study of which pairs (or n-uples) of states can be simultaneously connected thanks to an admissible control requires an additional controllability analysis with respect to the plain controllability mentioned above.

At the core of control design is then the notion of motion planning. Among the motion planning methods, a preeminent role is played by those based on the Lie algebra associated with the control system ([73], [60], [66]), those exploiting the possible flatness of the system ([47]) and those based on the continuation method ([86]). Optimal control is clearly another method for choosing a control law connecting two states, although it generally introduces new computational and theoretical difficulties.

Control systems with special structure, which are very important for applications are those for which the controls appear linearly. When the controls are not bounded, this means that the admissible velocities form a distribution in the tangent bundle to the state manifold. If the distribution is equipped with a smoothly varying norm (representing a cost of the control), the resulting geometrical structure is called *sub-Riemannian*. Sub-Riemannian geometry thus appears as the underlying geometry of the nonholonomic control systems, playing the same role as Euclidean geometry for linear systems. As such, its study is fundamental for control design. Moreover its importance goes far beyond control theory and is an active field of research both in differential geometry ([72]), geometric measure theory ([48], [21]) and hypoelliptic operator theory ([33]).

Other important classes of control systems are those modeling mechanical systems. The dynamics are naturally defined on the tangent or cotangent bundle of the configuration manifold, they have Lagrangian or Hamiltonian structure, and the controls act as forces. When the controls appear linearly, the resulting model can be seen somehow as a second-order sub-Riemannian structure (see [38]).

The control design topics presented above naturally extend to the case of distributed parameter control systems. The geometric approach to control systems governed by partial differential equations is a novel subject with great potential. It could complement purely analytical and numerical approaches, thanks to its more dynamical, qualitative and intrinsic point of view. An interesting example of this approach is the paper [18] about the controllability of Navier–Stokes equation by low forcing modes.

4. Application Domains

4.1. Quantum control

The issue of designing efficient transfers between different atomic or molecular levels is crucial in atomic and molecular physics, in particular because of its importance in those fields such as photochemistry (control by laser pulses of chemical reactions), nuclear magnetic resonance (NMR, control by a magnetic field of spin dynamics) and, on a more distant time horizon, the strategic domain of quantum computing. This last application explicitly relies on the design of quantum gates, each of them being, in essence, an open loop control law devoted to a prescribed simultaneous control action. NMR is one of the most promising techniques for the implementation of a quantum computer.

Physically, the control action is realized by exciting the quantum system by means of one or several external fields, being them magnetic or electric fields. The resulting control problem has attracted increasing attention, especially among quantum physicists and chemists (see, for instance, [78], [84]). The rapid evolution of the domain is driven by a multitude of experiments getting more and more precise and complex (see the recent review [37]). Control strategies have been proposed and implemented, both on numerical simulations and on physical systems, but there is still a large gap to fill before getting a complete picture of the control properties of quantum systems. Control techniques should necessarily be innovative, in order to take into account the physical peculiarities of the model and the specific experimental constraints.

The area where the picture got clearer is given by finite dimensional linear closed models.

- **Finite dimensional** refers to the dimension of the space of wave functions, and, accordingly, to the finite number of energy levels.
- **Linear** means that the evolution of the system for a fixed (constant in time) value of the control is determined by a linear vector field.
- **Closed** refers to the fact that the systems are assumed to be totally disconnected from the environment, resulting in the conservation of the norm of the wave function.

The resulting model is well suited for describing spin systems and also arises naturally when infinite dimensional quantum systems of the type discussed below are replaced by their finite dimensional Galerkin approximations. Without seeking exhaustiveness, let us mention some of the issues that have been tackled for finite dimensional linear closed quantum systems:

- controllability [19],
- bounds on the controllability time [15],
- STIRAP processes [89],
- simultaneous control [61],
- optimal control ([57], [28], [39]),
- numerical simulations [67].

Several of these results use suitable transformations or approximations (for instance the so-called rotating wave) to reformulate the finite-dimensional Schrödinger equation as a sub-Riemannian system. Open systems have also been the object of an intensive research activity (see, for instance, [20], [58], [79], [34]).

In the case where the state space is infinite dimensional, some optimal control results are known (see, for instance, [24], [35], [54], [25]). The controllability issue is less understood than in the finite dimensional setting, but several advances should be mentioned. First of all, it is known that one cannot expect exact controllability on the whole Hilbert sphere [88]. Moreover, it has been shown that a relevant model, the quantum oscillator, is not even approximately controllable [80], [70]. These negative results have been more recently completed by positive ones. In [26], [27] Beauchard and Coron obtained the first positive controllability result for a quantum particle in a 1D potential well. The result is highly nontrivial and is based on Coron's return method (see [43]). Exact controllability is proven to hold among regular enough wave functions. In particular, exact controllability among eigenfunctions of the uncontrolled Schrödinger operator can be achieved. Other important approximate controllability results have then been proved using Lyapunov methods [69], [74], [55]. While [69] studies a controlled Schrödinger equation in $\mathbb{R}$ for which the uncontrolled Schrödinger operator has mixed spectrum, [74], [55] deal mainly with general discrete-spectrum Schrödinger operators.

In all the positive results recalled in the previous paragraph, the quantum system is steered by a single external field. Different techniques can be applied in the case of two or more external fields, leading to additional controllability results [46], [31].

The picture is even less clear for nonlinear models, such as Gross–Pitaevski and Hartree–Fock equations. The obstructions to exact controllability, similar to the ones mentioned in the linear case, have been discussed in [52]. Optimal control approaches have also been considered [23], [36]. A comprehensive controllability analysis of such models is probably a long way away.

4.2. Neurophysiology

At the interface between neurosciences, mathematics, automatics and humanoid robotics, an entire new approach to neurophysiology is emerging. It arouses a strong interest in the four communities and its development requires a joint effort and the sharing of complementary tools.

A family of extremely interesting problems concerns the understanding of the mechanisms supervising some sensorial reactions or biomechanics actions such as image reconstruction by the primary visual cortex, eyes movement and body motion.

In order to study these phenomena, a promising approach consists in identifying the motion planning problems undertaken by the brain, through the analysis of the strategies that it applies when challenged by external inputs. The role of control is that of a language allowing to read and model neurological phenomena. The control algorithms would shed new light on the brain's geometric perception (the so-called neurogeometry [76]) and on the functional organization of the motor pathways.

- A challenging problem is that of the understanding of the mechanisms which are responsible for the process of image reconstruction in the primary visual cortex V1.

  The visual cortex areas composing V1 are notable for their complex spatial organization and their functional diversity. Understanding and describing their architecture requires sophisticated modeling tools. At the same time, the structure of the natural and artificial images used in visual psychophysics can be fully disclosed only using rather deep geometric concepts. The word “geometry" refers here to the internal geometry of the functional architecture of visual cortex areas (not to the geometry of the Euclidean external space). Differential geometry and analysis both play a fundamental role in the description of the structural characteristics of visual perception.

  A model of human perception based on a simplified description of the visual cortex V1, involving geometric objects typical of control theory and sub-Riemannian geometry, has been first proposed by Petitot ([77]) and then modified by Citti and Sarti ([42]). The model is based on experimental observations, and in particular on the fundamental work by Hubel and Wiesel [51] who received the Nobel prize in 1981.

  In this model, neurons of V1 are grouped into orientation columns, each of them being sensitive to visual stimuli arriving at a given point of the retina and oriented along a given direction. The retina is modeled by the real plane, while the directions at a given point are modeled by the projective line. The fiber bundle having as base the real plane and as fiber the projective line is called the *bundle of directions of the plane*.

  From the neurological point of view, orientation columns are in turn grouped into hypercolumns, each of them sensitive to stimuli arriving at a given point, oriented along any direction. In the same hypercolumn, relative to a point of the plane, we also find neurons that are sensitive to other stimuli properties, such as colors. Therefore, in this model the visual cortex treats an image not as a planar object, but as a set of points in the bundle of directions of the plane. The reconstruction is then realized by minimizing the energy necessary to activate orientation columns among those which are not activated directly by the image. This gives rise to a sub-Riemannian problem on the bundle of directions of the plane.

- Another class of challenging problems concern the functional organization of the motor pathways.

  The interest in establishing a model of the motor pathways, at the same time mathematically rigorous and biologically plausible, comes from the possible spillovers in neurophysiology. It could help to design better control strategies for robots and artificial limbs, rendering them capable to move more progressively and smoothly and also to react to exterior perturbations in a flexible way. An underlying relevant societal goal (clearly beyond our domain of expertise) is to clarify the mechanisms of certain debilitating troubles such as cerebellar disease, chorea and Parkinson's disease.

  A key issue in order to establish a model of the motor pathways is to determine the criteria underlying the brain's choices. For instance, for the problem of human locomotion (see [22]), identifying such criteria would be crucial to understand the neural pathways implicated in the generation of locomotion trajectories.

  A nowadays widely accepted paradigm is that, among all possible movements, the accomplished ones satisfy suitable optimality criteria (see [87] for a review). One is then led to study an inverse optimal control problem: starting from a database of experimentally recorded movements, identify a cost function such that the corresponding optimal solutions are compatible with the observed behaviors.

  Different methods have been taken into account in the literature to tackle this kind of problems, for instance in the linear quadratic case [56] or for Markov processes [75]. However all these methods have been conceived for very specific systems and they are not suitable in the general case. Two approaches are possible to overcome this difficulty. The direct approach consists in choosing a cost function among a class of functions naturally adapted to the dynamics (such as energy functions) and to compare the solutions of the corresponding optimal control problem to the experimental data. In particular one needs to compute, numerically or analytically, the optimal trajectories and to choose suitable criteria (quantitative and qualitative) for the comparison with observed trajectories. The inverse approach consists in deriving the cost function from the qualitative analysis of the data.

4.3. Switched systems

Switched systems form a subclass of hybrid systems, which themselves constitute a key growth area in automation and communication technologies with a broad range of applications. Existing and emerging areas include automotive and transportation industry, energy management and factory automation. The notion of hybrid systems provides a framework adapted to the description of the heterogeneous aspects related to the interaction of continuous dynamics (physical system) and discrete/logical components.

The characterizing feature of switched systems is the collective aspect of the dynamics. A typical question is that of stability, in which one wants to determine whether a dynamical system whose evolution is influenced by a time-dependent signal is uniformly stable with respect to all signals in a fixed class ([63]).

The theory of finite-dimensional hybrid and switched systems has been the subject of intensive research in the last decade and a large number of diverse and challenging problems such as stabilizability, observability, optimal control and synchronization have been investigated (see for instance [85], [64]).

The question of stability, in particular, because of its relevance for applications, has spurred a rich literature. Important contributions concern the notion of common Lyapunov function: when there exists a Lyapunov function that decays along all possible modes of the system (that is, for every possible constant value of the signal), then the system is uniformly asymptotically stable. Conversely, if the system is stable uniformly with respect to all signals switching in an arbitrary way, then a common Lyapunov function exists [65]. In the *linear* finite-dimensional case, the existence of a common Lyapunov function is actually equivalent to the global uniform exponential stability of the system [71] and, provided that the admissible modes are finitely many, the Lyapunov function can be taken polyhedral or polynomial [29], [30], [44]. A special role in the switched control literature has been played by common quadratic Lyapunov functions, since their existence can be tested rather efficiently (see [45] and references therein). Algebraic approaches to prove the stability of switched systems under arbitrary switching, not relying on Lyapunov techniques, have been proposed in [62], [16].

Other interesting issues concerning the stability of switched systems arise when, instead of considering arbitrary switching, one restricts the class of admissible signals, by imposing, for instance, a dwell time constraint [50].

Another rich area of research concerns discrete-time switched systems, where new intriguing phenomena appear, preventing the algebraic characterization of stability even for small dimensions of the state space [59]. It is known that, in this context, stability cannot be tested on periodic signals alone [32].

Finally, let us mention that little is known about infinite-dimensional switched system, with the exception of some results on uniform asymptotic stability ([68], [82], [83]) and some recent papers on optimal control ([49], [90]).

5. New Results

5.1. New results: geometric control

A first set of new results concerns sub-Riemannian geometry.

- In [3] we continued the study of almost-Riemannian structures, which are rank-varying sub-Riemannian structures locally generated by a number of vector fields equal to the dimension of the ambient manifold. In particular, two-dimensional almost-Riemannian structures are generalized Riemannian structures on surfaces for which local orthonormal frames are Lie bracket generating pair of vector fields that can become collinear. We considered the Carnot–Carathéodory distance canonically associated with an almost-Riemannian structure and studied the problem of Lipschitz equivalence between two such distances on a given compact oriented surface. We analyzed the generic case, allowing in particular for the presence of tangency points, i.e., points where two generators of the distribution and their Lie bracket are linearly dependent. The main result of the paper provides a characterization of the Lipschitz equivalence class of an almost-Riemannian distance in terms of a labeled graph associated with it.
- In [1] we studied nilpotent 2-step, corank 2 sub-Riemannian metrics. Such metrics naturally appear as nilpotent approximations of general sub-Riemannian ones. We exhibited optimal syntheses for these problems. It turns out that in general the cut time is not equal to the first conjugate time but has a simple explicit expression. As a byproduct of this study we proved some smoothness properties of the spherical Hausdorff measure in the case of a generic 6-dimensional, 2-step corank 2 sub-Riemannian metric.
- In [12] we started from the remark that in Carnot–Carathéodory spaces the class of 1-rectifiable sets does not contain smooth non-horizontal curves. We were looking for a new definition of rectifiable sets including non-horizontal curves. We introduced, for any metric space, a new class of curves, called continuously metric differentiable of degree k, which are Hölder but not Lipschitz continuous when $k > 1$. Replacing Lipschitz curves by this kind of curves we defined $(\mathcal{H}^k, 1)$-rectifiable sets and showed a density result generalizing the corresponding one in Euclidean geometry. This theorem has been obtained as a consequence of computations of Hausdorff measures along curves, for which we gave an integral formula. In particular, we showed that both spherical and standard Hausdorff measures along curves coincide with a class of dimensioned lengths and are related with an interpolation complexity, for which estimates have already been obtained in Carnot–Carathéodory spaces.

A class of problems for which tracking and motion planning is crucial, is given by the control of unmanned aerial vehicles (UAV). In order to develop improved planning tasks that take into account payload requirements, optimal costs and obstacles avoidance (or no flight zones), it is important to develop reliable and flexible simulators. One such simulator for a UAV ground control station is proposed in [9]. The research focuses on the connection between the UAV trajectories and its sensors. Our proposal includes a module-based description of the architecture of the simulator and is based on a nonlinear model of a fixed wing aircraft.

5.2. New results: quantum control

New results have been obtained for the control of the bilinear Schrödinger equation, with two different approaches.

- In [2] we proved an approximate controllability result by finite-dimensional methods, considering the Galerkin approximations. The approach improves the technique that we developed in [40]. The result requires less restrictive non-resonance hypotheses on the spectrum of the uncontrolled Schrödinger operator than those already known. The control operator is not required to be bounded and we are able to extend the controllability result to the density matrices. The proof is based on fine controllability properties of the finite-dimensional Galerkin approximations and allows to get estimates for the L^1 norm of the control. The general controllability result is applied to the problem of controlling the rotation of a bipolar rigid molecule confined on a plane by means of two orthogonal external fields.
- In [4] we presented a constructive method to control the bilinear Schrödinger equation via two controls. The method is based on adiabatic theory and works if the spectrum of the Hamiltonian admits conical eigenvalue intersections. We provided sharp estimates of the relation between the error and the controllability time. We also showed that for a Hamiltonian of the kind $-\Delta + V_0(x) + u_1 V_1(x) + u_2 V_2(x)$ on a domain of $\mathbb{R}^n$ the eigenvalue intersections are conical generically with respect to V_0, V_1, V_2.

5.3. New results: neurophysiology

We gave new contributions to the developing theory of human locomotion modeled through optimal control problems. In this paradigm, the trajectories are assumed to be solutions of an optimal control problem whose cost has to be determined.

- The purpose of [6] has been to analyze the class of optimal control problems defined in this way. We proved strong convergence of their solutions, on the one hand for perturbations of the initial and final points (stability), and on the other hand for perturbations of the cost (robustness).
- In [5] we discussed the modeling of both the dynamical system and the cost to be minimized, and we analyzed the corresponding optimal synthesis. The main results describe the asymptotic behavior of the optimal trajectories as the target point goes to infinity.

In [10] we studied the model of geometry of vision due to Petitot, Citti and Sarti [81]. One of the main features of this model is that the primary visual cortex V1 lifts an image from $\mathbb{R}^2$ to the bundle of directions of the plane. Neurons are grouped into orientation columns, each of them corresponding to a point of this bundle. In this model a corrupted image is reconstructed by minimizing the energy necessary for the activation of the orientation columns corresponding to regions in which the image is corrupted. The minimization process intrinsically defines an hypoelliptic heat equation on the bundle of directions of the plane. In the original model, directions are considered both with and without orientation giving rise respectively to a problem on the group of rototranslations of the plane $SE(2)$ or on the projective tangent bundle of the plane. We provided a mathematical proof of several important facts for this model. We first proved that the model is mathematically consistent only if directions are considered without orientation. We then proved that the convolution of a $L^2(\mathbb{R}^2, \mathbb{R})$ function (e.g. an image) with a 2D Gaussian is generically a Morse function. This fact is important since the lift of Morse functions to the projective tangent bundle of the plane is defined on a smooth manifold.

We then provided the explicit expression of the hypoelliptic heat kernel on the projective tangent bundle of the plane in terms of Mathieu functions. Finally, we presented the main ideas of an algorithm which allows to perform image reconstruction on real non-academic images. The algorithm is massively parallelizable and needs no information on where the image is corrupted.

5.4. New results: switched systems

New results on switched systems have been obtained in three directions:

- **Discrete-time systems.** In [14] we dealt with the stability properties of linear discrete-time switched systems with polytopic sets of modes. The most classical and viable way of studying the uniform asymptotic stability of such a system is to check for the existence of a quadratic Lyapunov function. It is known from the literature that letting the Lyapunov function depend on the time-varying switching parameter improves the chance that a quadratic Lyapunov function exists. The contribution of [14] is twofold. We first proved that under a non-degeneracy assumption the dependence on the switching function can be actually assumed to be linear with no prejudice on the effectiveness of the method. Moreover, we showed that no gain is obtained even if we allow the Lyapunov function to depend on the time. Second, we introduced the notion of eventual accessible sets and we showed that, in the degenerate case, it leads to a relaxation of the LMI conditions to check stability of switched linear systems. As a consequence, equivalence between different notions of quadratic stability can still be established under an additional assumption but, in general, allowing the Lyapunov function to depend on time leads to less conservative LMI conditions, as we explicitly showed through an example. We also discussed the case where the variation of the switching parameter is bounded by a prescribed constant between two subsequent times.
- **Continuous-time systems subject to persistent-excitation.** In [11] we studied linear control systems for which the controlled part can be switched off by a signal subject to a persistent excitation condition. We were interested in the stabilization problem of this system by a linear state feedback and we positively answered a question asked in [41], proving the following: Assume that the class of persistently exciting signals is restricted to those which are M-Lipschitzian, where $M > 0$ is a positive constant. Then, given any $C > 0$, there exists a linear state feedback depending on the class of signals under consideration (but not an individual signal) so that the rate of exponential decay of the time-varying system associated with any signal is greater than C.
- **Infinite-dimensional continuous-time systems.** In [13] we partially extended the analysis of finite-dimensional systems subject to persistently exciting signals to the case of systems driven by PDEs. More precisely, we studied the asymptotic stability of a dissipative evolution in a Hilbert space subject to intermittent damping. We observed that, even if the intermittence satisfies a persistent excitation condition, if the Hilbert space is infinite-dimensional then the system needs not being asymptotically stable (not even in the weak sense). Exponential stability is recovered under a generalized observability inequality, allowing for time-domains that are not intervals. Weak asymptotic stability is obtained under a similarly generalized unique continuation principle. Strong asymptotic stability is proved for intermittences that do not necessarily satisfy some persistent excitation condition, evaluating their total contribution to the decay of the trajectories of the damped system. Our results are discussed using the example of the wave equation and the linear Schrödinger equation.

6. Partnerships and Cooperations

6.1. Regional Initiatives

- **Digitéo project CONGEO.** CONGEO (2009–2013) is financed by Digitéo in the framework of the DIM *Logiciels et systèmes complexes*. It focuses on the neurophysiology applications. U. Boscain, Y. Chitour (leader), F. Jean and P. Mason are part of the project.

6.2. National Initiatives

- **ANR project GCM.** The project ANR GCM (*programme blanc*, 2009–13) involves the great majority of GECO's members (permanent and external). It focuses on various theoretical aspects of geometric control and on quantum control. It is coordinated by J.-P. Gauthier.
- **ANR ArHyCo.** The project ANR ArHyCo (*programme ARPEGE*, 2009–13) is about switched systems. It is coordinated by J. Daafouz. The first theme of the ANR, on stability of switched systems, is lead by M. Sigalotti.

6.3. European Initiatives

6.3.1. Collaborations in European Programs

Program: ERC Starting Grant

Project acronym: GeCoMethods

Project title: Geometric Control Methods for the Heat and Schroedinger Equations

Duration: 1/5/2010 - 1/5/2015

Coordinator: Ugo Boscain

Abstract: The aim of this project is to study certain PDEs for which geometric control techniques open new horizons. More precisely we plan to exploit the relation between the sub-Riemannian distance and the properties of the kernel of the corresponding hypoelliptic heat equation and to study controllability properties of the Schroedinger equation.

All subjects studied in this project are applications-driven: the problem of controllability of the Schroedinger equation has direct applications in Laser spectroscopy and in Nuclear Magnetic Resonance; the problem of nonisotropic diffusion has applications in cognitive neuroscience (in particular for models of human vision).

Participants. Main collaborator: Mario Sigalotti. Other members of the team: Andrei Agrachev, Riccardo Adami, Thomas Chambrion, Grégoire Charlot, Yacine Chitour, Jean-Paul Gauthier, Frédéric Jean.

6.3.2. Major European Organizations with which you have followed Collaborations

SISSA (Scuola Internazionale Superiore di Studi Avanzati), Trieste, Italy.

Sector of Functional Analysis and Applications, Geometric Control group. Coordinator: Andrei A. Agrachev.

We collaborate with the Geometric Control group at SISSA mainly on subjects related with sub-Riemannian geometry. Thanks partly to our collaboration, SISSA has established an official research partnership with École Polytechnique.

6.4. International Initiatives

6.4.1. Visits of International Scientists

Remco Duits, Eindhoven University of Technology. June 2011

6.4.2. Participation In International Programs

- Laboratoire Euro Maghrébin de Mathématiques et de leurs Interactions (LEM2I)
  http://www.lem2i.cnrs.fr/
- GDRE Control of Partial Differential Equations (CONEDP)
  http://www.ceremade.dauphine.fr/~glass/GDRE/

7. Dissemination

7.1. Animation of the scientific community

Conference organization

- “Control and Topology”, Topolò, Italy, 29-31 May 2011 (organizers: U. Boscain, Y. Chitour, M. Sigalotti).
- IFIP 2011, Double session on “Optimization and Control of Nanosystems I-II’, September 2011 (organizers: A. Borzi, U. Boscain)
- IFIP 2011, Double session on “Analytic and Geometric Optimal Control I-II”, September 2011(organizers: U. Boscain, J.B. Caillau).

Editorial activity

- U. Boscain is Associate Editor of Journal of Dynamical and Control Systems, ESAIM Control, Optimisation and Calculus of Variations, Mathematical Control and Related Fields. He is also referee for Journal of Differential equations, AIMS Book series: Applied mathematics, SIAM J. Control Optim., Automatica, Rendiconti dei Lincei, Matematica ed Applicazioni, Physica A...and for the conferences ACC, CDC, MTNS...
- M. Sigalotti is Referee for IEEE TAC, SIAM J. Control Optim., Automatica, MathSciNet, Journal of Functional Analysis...and for the conferences CDC, ACC, IFAC...

7.2. Teaching

Licence: Mario Sigalotti, Systèmes Dynamiques: Stabilité et Commande (main teacher: F. Jean), 16 hours TD, first year engineering school (L3), ENSTA

PhD: Ugo Boscain, Introduction to geodesics in sub-Riemannian geometry, 5 hours, Seventh School on Analysis and Geometry in Metric Spaces, Levico Terme, Italy.

PhD & HdR

PhD in progress: Dario Prandi, “Geometric control and PDEs", 1/9/2011, supervisors: Ugo Boscain, Mario Sigalotti.

PhD in progress: Moussa Gaye, “Some problems of geometric analysis in almost-Riemannian geometry and of stability of switching systems", 1/9/2011, supervisors: Ugo Boscain, Paolo Mason.

8. Bibliography

Publications of the year

Articles in International Peer-Reviewed Journal

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[2] U. BOSCAIN, M. CAPONIGRO, T. CHAMBRION, M. SIGALOTTI. *A weak spectral condition for the controllability of the bilinear Schrödinger equation with application to the control of a rotating planar molecule*, in "Communications in Mathematical Physics", 2012, to appear.

[3] U. BOSCAIN, G. CHARLOT, R. GHEZZI, M. SIGALOTTI. *Lipschitz classification of almost-Riemannian distances on compact oriented surfaces* , in "Journal of Geometric Analysis", 2012, to appear, http://www.springerlink.com/content/3771184321200437/.

[4] U. BOSCAIN, F. CHITTARO, P. MASON, M. SIGALOTTI. *Adiabatic control of the Schrödinger equation via conical intersections of the eigenvalues*, in "IEEE Trans. Automat. Control", 2012, to appear.

[5] Y. CHITOUR, F. JEAN, P. MASON. *Optimal control models of the goal-oriented human locomotion*, in "SIAM J. Control Optim.", 2012, to appear.

[6] F. CHITTARO, F. JEAN, P. MASON. *On the inverse optimal control problems of the human locomotion: stability and robustness of the minimizers*, in "Journal of Mathematical Sciences", 2011.

[7] F. LAFONT, E. BUSVELLE, J.-P. GAUTHIER. *An adaptive high-gain observer for wastewater treatment systems*, in "Journal of Process Control", 2011, p. 893–900.

[8] S. METHNANI, J.-P. GAUTHIER, F. LAFONT. *Sensor fault reconstruction and observability for unknown inputs, with an application to wastewater treatment plants*, in "International Journal of Control", 2011, http://www.informaworld.com/smpp/title~content=t713393989.

National Conferences with Proceeding

[9] A. AJAMI, T. MAILLOT, N. BOIZOT, J.-F. BALMAT, J.-P. GAUTHIER. *Simulation of a UAV ground control station*, in "Performance, interoperability and safety for sustainable development", 2012.

Other Publications

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