

Activity Report 2011

Team **REGULARITY**

Probabilistic modelling of irregularity and application to uncertainties management

RESEARCH CENTER
Saclay - Île-de-France

THEME
Stochastic Methods and Models

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2. Overall Objectives

2.1. Overall Objectives

Many phenomena of interest are analyzed and controlled through graphs or n-dimensional images. Often, these graphs have an *irregular aspect*, whether the studied phenomenon is of natural or artificial origin. In the first class, one may cite natural landscapes, most biological signals and images (EEG, ECG, MR images, ...), and temperature records. In the second class, prominent examples include financial logs and TCP traces.

Such irregular phenomena are usually not adequately described by purely deterministic models, and a probabilistic ingredient is often added. Stochastic processes allow to take into account, with a firm theoretical basis, the numerous microscopic fluctuations that shape the phenomenon.

In general, it is a wrong view to believe that irregularity appears as an epiphenomenon, that is conveniently dealt with by introducing randomness. In many situations, and in particular in some of the examples mentioned above, irregularity is a core ingredient that cannot be removed without destroying the phenomenon itself. In some cases, irregularity is even a necessary condition for proper functioning. A striking example is that of ECG: an ECG is inherently irregular, and, moreover, in a mathematically precise sense, an *increase* in its regularity is strongly correlated with a *degradation* of its condition.

In fact, in various situations, irregularity is a crucial feature that can be used to assess the behaviour of a given system. For instance, irregularity may be the result of two or more sub-systems that act in a concurrent way to achieve some kind of equilibrium. Examples of this abound in nature (*e.g.* the sympathetic and parasympathetic systems in the regulation of the heart). For artifacts, such as financial logs and TCP traffic, irregularity is in a sense an unwanted feature, since it typically makes regulations more complex. It is again, however, a necessary one. For instance, efficiency in financial markets requires a constant flow of information among agents, which manifests itself through permanent fluctuations of the prices: irregularity just reflects the evolution of this information.

The aim of *Regularity* is to develop a coherent set of methods allowing to model such “essentially irregular” phenomena in view of managing the uncertainties entailed by their irregularity.

Indeed, essential irregularity makes it more difficult to study phenomena in terms of their description, modeling, prediction and control. It introduces *uncertainties* both in the measurements and the dynamics. It is, for instance, obviously easier to predict the short time behaviour of a smooth (e.g. C^1) process than of a nowhere differentiable one. Likewise, sampling rough functions yields less precise information than regular ones. As a consequence, when dealing with essentially irregular phenomena, uncertainties are fundamental in the sense that one cannot hope to remove them by a more careful analysis or a more adequate modeling. The study of such phenomena then requires to develop specific approaches allowing to manage in an efficient way these inherent uncertainties.

3. Scientific Foundations

3.1. Theoretical aspects: probabilistic modeling of irregularity

The modeling of essentially irregular phenomena is an important challenge, with an emphasis on understanding the sources and functions of this irregularity. Probabilistic tools are well-adapted to this task, provided one can design stochastic models for which the regularity can be measured and controlled precisely. Two points deserve special attention:

- first, the study of regularity has to be *local*. Indeed, in most applications, one will want to act on a system based on local temporal or spatial information. For instance, detection of arrhythmias in ECG or of krachs in financial markets should be performed in “real time”, or, even better, ahead of time. In this sense, regularity is a *local* indicator of the *local* health of a system.
- Second, although we have used the term “irregularity” in a generic and somewhat vague sense, it seems obvious that, in real-world phenomena, regularity comes in many colors, and a rigorous analysis should distinguish between them. As an example, at least two kinds of irregularities are present in financial logs: the local “roughness” of the records, and the local density and height of jumps. These correspond to two different concepts of regularity (in technical terms, Hölder exponents and local index of stability), and they both contribute a different manner to financial risk.

In view of the above, the *Regularity* team focuses on the design of methods that:

1. define and study precisely various relevant measures of local regularity,
2. allow to build stochastic models versatile enough to mimic the rapid variations of the different kinds of regularities observed in real phenomena,
3. allow to estimate as precisely and rapidly as possible these regularities, so as to alert systems in charge of control.

Our aim is to address the three items above through the design of mathematical tools in the field of probability (and, to a lesser extent, statistics), and to apply these tools to uncertainty management as described in the following section. We note here that we do not intend to address the problem of controlling the phenomena based on regularity, that would naturally constitute an item 4 in the list above. Indeed, while we strongly believe that generic tools may be designed to measure and model regularity, and that these tools may be used to analyze real-world applications, in particular in the field of uncertainty management, it is clear that, when it comes to control, application-specific tools are required, that we do not wish to address.

The research topics of the *Regularity* team can be roughly divided into two strongly interacting axes, corresponding to two complementary ways of studying regularity:

1. developments of tools allowing to characterize, measure and estimate various notions of local regularity, with a particular emphasis on the stochastic frame,
2. definition and fine analysis of stochastic models for which some aspects of local regularity may be prescribed.

These two aspects are detailed in sections 3.2 and 3.3 below.

3.2. Tools for characterizing and measuring regularity

Fractional Dimensions

Although the main focus of our team is on characterizing *local* regularity, on occasions, it is interesting to use a *global* index of regularity. Fractional dimensions provide such an index. In particular, the *regularization dimension*, that was defined in [35], is well adapted to the study stochastic processes, as its definition allows to build robust estimators in an easy way. Since its introduction, regularization dimension has been used by various teams worldwide in many different applications including the characterization of certain stochastic processes, statistical estimation, the study of mammographies or galactograms for breast carcinomas detection, ECG analysis for the study of ventricular arrhythmia, encephalitis diagnosis from EEG, human skin analysis, discrimination between the nature of radioactive contaminations, analysis of porous media textures, well-logs data analysis, agro-alimentary image analysis, road profile analysis, remote sensing, mechanical systems assessment, analysis of video games, ... (see <http://regularity.saclay.inria.fr/theory/localregularity/biblioregdim> for a list of works using the regularization dimension).

Hölder exponents

The simplest and most popular measures of local regularity are the pointwise and local Hölder exponents. For a stochastic process $\{X(t)\}_{t \in \mathbb{R}}$ whose trajectories are continuous and nowhere differentiable, these are defined, at a point t_0 , as the random variables:

$$\alpha_X(t_0, \omega) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{t, u \in B(t_0, \rho)} \frac{|X_t - X_u|}{\rho^\alpha} < \infty \right\}, \quad (1)$$

and

$$\tilde{\alpha}_X(t_0, \omega) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{t, u \in B(t_0, \rho)} \frac{|X_t - X_u|}{\|t - u\|^\alpha} < \infty \right\}. \quad (2)$$

Although these quantities are in general random, we will omit as is customary the dependency in ω and X and write $\alpha(t_0)$ and $\tilde{\alpha}(t_0)$ instead of $\alpha_X(t_0, \omega)$ and $\tilde{\alpha}_X(t_0, \omega)$.

The random functions $t \mapsto \alpha_X(t_0, \omega)$ and $t \mapsto \tilde{\alpha}_X(t_0, \omega)$ are called respectively the pointwise and local Hölder functions of the process X .

The pointwise Hölder exponent is a very versatile tool, in the sense that the set of pointwise Hölder functions of continuous functions is quite large (it coincides with the set of lower limits of sequences of continuous functions [7]). In this sense, the pointwise exponent is often a more precise tool (*i.e.* it varies in a more rapid way) than the local one, since local Hölder functions are always lower semi-continuous. This is why, in particular, it is the exponent that is used as a basis ingredient in multifractal analysis (see section 3.2). For certain classes of stochastic processes, and most notably Gaussian processes, it has the remarkable property that, at each point, it assumes an almost sure value [19]. SRP, mBm, and processes of this kind (see sections 3.3 and 3.3) rely on the sole use of the pointwise Hölder exponent for prescribing the regularity.

However, α_X obviously does not give a complete description of local regularity, even for continuous processes. It is for instance insensitive to “oscillations”, contrarily to the local exponent. A simple example in the deterministic frame is provided by the function $x^\gamma \sin(x^{-\beta})$, where γ, β are positive real numbers. This so-called “chirp function” exhibits two kinds of irregularities: the first one, due to the term x^γ is measured by the pointwise Hölder exponent. Indeed, $\alpha(0) = \gamma$. The second one is due to the wild oscillations around 0, to which α is blind. In contrast, the local Hölder exponent at 0 is equal to $\frac{\gamma}{1+\beta}$, and is thus influenced by the oscillatory behaviour.

Another, related, drawback of the pointwise exponent is that it is not stable under integro-differentiation, which sometimes makes its use complicated in applications. Again, the local exponent provides here a useful complement to α , since $\tilde{\alpha}$ is stable under integro-differentiation.

Both exponents have proved useful in various applications, ranging from image denoising and segmentation to TCP traffic characterization. Applications require precise estimation of these exponents.

Stochastic 2-microlocal analysis

Neither the pointwise nor the local exponents give a complete characterization of the local regularity, and, although their joint use somewhat improves the situation, it is far from yielding the complete picture.

A fuller description of local regularity is provided by the so-called *2-microlocal analysis*, introduced by J.M. Bony [44]. In this frame, regularity at each point is now specified by two indices, which makes the analysis and estimation tasks more difficult. More precisely, a function f is said to belong to the *2-microlocal space* $C_{x_0}^{s,s'}$, where $s + s' > 0$, $s' < 0$, if and only if its $m = [s + s']$ -th order derivative exists around x_0 , and if there exists $\delta > 0$, a polynomial P with degree lower than $[s] - m$, and a constant C , such that

$$\left| \frac{\partial^m f(x) - P(x)}{|x - x_0|^{[s] - m}} - \frac{\partial^m f(y) - P(y)}{|y - x_0|^{[s] - m}} \right| \leq C |x - y|^{s + s' - m} (|x - y| + |x - x_0|)^{-s' - [s] + m}$$

for all x, y such that $0 < |x - x_0| < \delta$, $0 < |y - x_0| < \delta$. This characterization was obtained in [26], [36]. See [56], [57] for other characterizations and results. These spaces are stable through integro-differentiation, i.e. $f \in C_x^{s,s'}$ if and only if $f' \in C_x^{s-1,s'}$. Knowing to which space f belongs thus allows to predict the evolution of its regularity after derivation, a useful feature if one uses models based on some kind differential equations. A lot of work remains to be done in this area, in order to obtain more general characterizations, to develop robust estimation methods, and to extend the “2-microlocal formalism” : this is a tool allowing to detect which space a function belongs to, from the computation of the Legendre transform of an auxiliary function known as its *2-microlocal spectrum*. This spectrum provide a wealth of information on the local regularity.

In [19], we have laid some foundations for a stochastic version of 2-microlocal analysis. We believe this will provide a fine analysis of the local regularity of random processes in a direction different from the one detailed for instance in [62]. We have defined random versions of the 2-microlocal spaces, and given almost sure conditions for continuous processes to belong to such spaces. More precise results have also been obtained for Gaussian processes. A preliminary investigation of the 2-microlocal behaviour of Wiener integrals has been performed.

Multifractal analysis of stochastic processes

A direct use of the local regularity is often fruitful in applications. This is for instance the case in RR analysis or terrain modeling. However, in some situations, it is interesting to supplement or replace it by a more global approach known as *multifractal analysis* (MA). The idea behind MA is to group together all points with same regularity (as measured by the pointwise Hölder exponent) and to measure the “size” of the sets thus obtained [32], [45], [52]. There are mainly two ways to do so, a geometrical and a statistical one.

In the geometrical approach, one defines the *Hausdorff multifractal spectrum* of a process or function X as the function: $\alpha \mapsto f_h(\alpha) = \dim \{t : \alpha_X(t) = \alpha\}$, where $\dim E$ denotes the Hausdorff dimension of the set E . This gives a fine measure-theoretic information, but is often difficult to compute theoretically, and almost impossible to estimate on numerical data.

The statistical path to MA is based on the so-called *large deviation multifractal spectrum*:

$$f_g(\alpha) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{\log N_n^\varepsilon(\alpha)}{\log n},$$

where:

$$N_n^\varepsilon(\alpha) = \#\{k : \alpha - \varepsilon \leq \alpha_n^k \leq \alpha + \varepsilon\},$$

and α_n^k is the “coarse grained exponent” corresponding to the interval $I_n^k = [\frac{k}{n}, \frac{k+1}{n}]$, i.e.:

$$\alpha_n^k = \frac{\log |Y_n^k|}{-\log n}.$$

Here, Y_n^k is some quantity that measures the variation of X in the interval I_n^k , such as the increment, the oscillation or a wavelet coefficient.

The large deviation spectrum is typically easier to compute and to estimate than the Hausdorff one. In addition, it often gives more relevant information in applications.

Under very mild conditions (e.g. for instance, if the support of f_g is bounded, [41]) the concave envelope of f_g can be computed easily from an auxiliary function, called the *Legendre multifractal spectrum*. To do so, one basically interprets the spectrum f_g as a rate function in a large deviation principle (LDP): define, for $q \in \mathbb{R}$,

$$S_n(q) = \sum_{k=0}^{n-1} |Y_n^k|^q, \quad (3)$$

with the convention $0^q := 0$ for all $q \in \mathbb{R}$. Let:

$$\tau(q) = \liminf_{n \rightarrow \infty} \frac{\log S_n(q)}{-\log(n)}.$$

The Legendre multifractal spectrum of X is defined as the Legendre transform τ^* of τ :

$$f_l(\alpha) := \tau^*(\alpha) := \inf_{q \in \mathbb{R}} (q\alpha - \tau(q)).$$

To see the relation between f_g and f_l , define the sequence of random variables $Z_n := \log |Y_n^k|$ where the randomness is through a choice of k uniformly in $\{0, \dots, n-1\}$. Consider the corresponding moment generating functions:

$$c_n(q) := -\frac{\log E_n[\exp(qZ_n)]}{\log(n)}$$

where E_n denotes expectation with respect to P_n , the uniform distribution on $\{0, \dots, n-1\}$. A version of Gärtner-Ellis theorem ensures that if $\lim c_n(q)$ exists (in which case it equals $1 + \tau(q)$), and is differentiable, then $c^* = f_g - 1$. In this case, one says that the *weak multifractal formalism* holds, i.e. $f_g = f_l$. In favorable cases, this also coincides with f_h , a situation referred to as the *strong multifractal formalism*.

Multifractal spectra subsume a lot of information about the distribution of the regularity, that has proved useful in various situations. A most notable example is the strong correlation reported recently in several works between the narrowing of the multifractal spectrum of ECG and certain pathologies of the heart [53], [55]. Let us also mention the multifractality of TCP traffic, that has been both observed experimentally and proved on simplified models of TCP [2], [42].

Another colour in local regularity: jumps

As noted above, apart from Hölder exponents and their generalizations, at least another type of irregularity may sometimes be observed on certain real phenomena: discontinuities, which occur for instance on financial logs and certain biomedical signals. In this frame, it is of interest to supplement Hölder exponents and their extensions with (at least) an additional index that measures the local intensity and size of jumps. This is a topic we intend to pursue in full generality in the near future. So far, we have developed an approach in the particular frame of *multistable processes*. We refer to section 3.3 for more details.

3.3. Stochastic models

The second axis in the theoretical developments of the *Regularity* team aims at defining and studying stochastic processes for which various aspects of the local regularity may be prescribed.

Multifractional Brownian motion

One of the simplest stochastic process for which some kind of control over the Hölder exponents is possible is probably fractional Brownian motion (fBm). This process was defined by Kolmogorov and further studied by Mandelbrot and Van Ness, followed by many authors. The so-called “moving average” definition of fBm reads as follows:

$$Y_t = \int_{-\infty}^0 \left[(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right] \cdot \mathbb{W}(du) + \int_0^t (t-u)^{H-\frac{1}{2}} \cdot \mathbb{W}(du),$$

where \mathbb{W} denotes the real white noise. The parameter H ranges in $(0, 1)$, and it governs the pointwise regularity: indeed, almost surely, at each point, both the local and pointwise Hölder exponents are equal to H .

Although varying H yields processes with different regularity, the fact that the exponents are constant along any single path is often a major drawback for the modeling of real world phenomena. For instance, fBm has often been used for the synthesis natural terrains. This is not satisfactory since it yields images lacking crucial features of real mountains, where some parts are smoother than others, due, for instance, to erosion.

It is possible to generalize fBm to obtain a Gaussian process for which the pointwise Hölder exponent may be tuned at each point: the *multifractional Brownian motion (mBm)* is such an extension, obtained by substituting the constant parameter $H \in (0, 1)$ with a *regularity function* $H : \mathbb{R}_+ \rightarrow (0, 1)$.

mBm was introduced independently by two groups of authors: on the one hand, Peltier and Levy-Vehel [33] defined the mBm $\{X_t; t \in \mathbb{R}_+\}$ from the moving average definition of the fractional Brownian motion, and set:

$$X_t = \int_{-\infty}^0 \left[(t-u)^{H(t)-\frac{1}{2}} - (-u)^{H(t)-\frac{1}{2}} \right] \cdot \mathbb{W}(du) + \int_0^t (t-u)^{H(t)-\frac{1}{2}} \cdot \mathbb{W}(du),$$

On the other hand, Benassi, Jaffard and Roux [43] defined the mBm from the harmonizable representation of the fBm, *i.e.*:

$$X_t = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+\frac{1}{2}}} \cdot \widehat{\mathbb{W}}(d\xi),$$

where $\widehat{\mathbb{W}}$ denotes the complex white noise.

The Hölder exponents of the mBm are prescribed almost surely: the pointwise Hölder exponent is $\alpha_X(t) = H(t) \wedge \alpha_H(t)$ a.s., and the local Hölder exponent is $\tilde{\alpha}_X(t) = H(t) \wedge \tilde{\alpha}_H(t)$ a.s. Consequently, the regularity of the sample paths of the mBm are determined by the function H or by its regularity. The multifractional Brownian motion is our prime example of a stochastic process with prescribed local regularity.

The fact that the local regularity of mBm may be tuned *via* a functional parameter has made it a useful model in various areas such as finance, biomedicine, geophysics, image analysis, A large number of studies have been devoted worldwide to its mathematical properties, including in particular its local time. In addition, there is now a rather strong body of work dealing the estimation of its functional parameter, *i.e.* its local regularity. See <http://regularity.saclay.inria.fr/theory/stochasticmodels/bibliombm> for a partial list of works, applied or theoretical, that deal with mBm.

Self-regulating processes

We have recently introduced another class of stochastic models, inspired by mBm, but where the local regularity, instead of being tuned “exogenously”, is a function of the amplitude. In other words, at each point t , the Hölder exponent of the process X verifies almost surely $\alpha_X(t) = g(X(t))$, where g is a fixed deterministic function verifying certain conditions. A process satisfying such an equation is generically termed a *self-regulating process* (SRP). The particular process obtained by adapting adequately mBm is called the self-regulating multifractional process [3]. Another instance is given by modifying the LÄ©vy construction of Brownian motion [39]. The motivation for introducing self-regulating processes is based on the following general fact: in nature, the local regularity of a phenomenon is often related to its amplitude. An intuitive example is provided by natural terrains: in young mountains, regions at higher altitudes are typically more irregular than regions at lower altitudes. We have verified this fact experimentally on several digital elevation models [9]. Other natural phenomena displaying a relation between amplitude and exponent include temperatures records and RR intervals extracted from ECG [39].

To build the SRMP, one starts from a field of fractional Brownian motions $B(t, H)$, where (t, H) span $[0, 1] \times [a, b]$ and $0 < a < b < 1$. For each fixed H , $B(t, H)$ is a fractional Brownian motion with exponent H . Denote:

$$\overline{X}_{\alpha'}^{\beta'} = \alpha' + (\beta' - \alpha') \frac{X - \min_K(X)}{\max_K(X) - \min_K(X)}$$

the affine rescaling between α' and β' of an arbitrary continuous random field over a compact set K . One considers the following (stochastic) operator, defined almost surely:

$$\begin{aligned} \Lambda_{\alpha', \beta'} : \mathcal{C}([0, 1], [\alpha, \beta]) &\rightarrow \mathcal{C}([0, 1], [\alpha, \beta]) \\ Z(\cdot) &\mapsto \overline{B(\cdot, g(Z(\cdot)))}_{\alpha'}^{\beta'} \end{aligned}$$

where $\alpha \leq \alpha' < \beta' \leq \beta$, α and β are two real numbers, and α', β' are random variables adequately chosen. One may show that this operator is contractive with respect to the sup-norm. Its unique fixed point is the SRMP. Additional arguments allow to prove that, indeed, the HÄ©lder exponent at each point is almost surely $g(t)$.

An example of a two dimensional SRMP with function $g(x) = 1 - x^2$ is displayed on figure 1.

We believe that SRP open a whole new and very promising area of research.

Multistable processes

Non-continuous phenomena are commonly encountered in real-world applications, *e.g.* financial records or EEG traces. For such processes, the information brought by the HÄ©lder exponent must be supplemented by some measure of the density and size of jumps. Stochastic processes with jumps, and in particular LÄ©vy processes, are currently an active area of research.

The simplest class of non-continuous LÄ©vy processes is maybe the one of stable processes [64]. These are mainly characterized by a parameter $\alpha \in (0, 2]$, the *stability index* ($\alpha = 2$ corresponds to the Gaussian case, that we do not consider here). This index measures in some precise sense the intensity of jumps. Paths of stable processes with α close to 2 tend to display “small jumps”, while, when α is near 0, their aspect is governed by large ones.

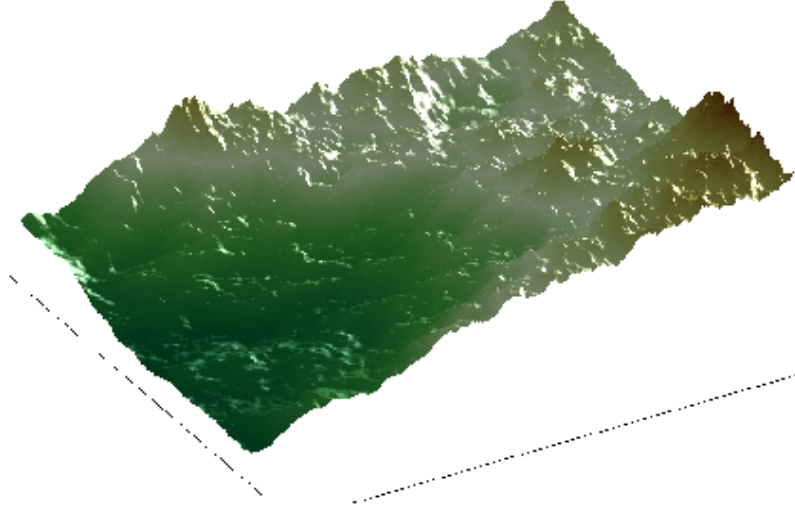


Figure 1. Self-regulating multifractional process with $g(x) = 1 - x^2$

In line with our quest for the characterization and modeling of various notions of local regularity, we have defined *multistable processes*. These are processes which are “locally” stable, but where the stability index α is now a function of time. This allows to model phenomena which, at times, are “almost continuous”, and at others display large discontinuities. Such a behaviour is for instance obvious on almost any sufficiently long financial record.

More formally, a multistable process is a process which is, at each time u , tangent to a stable process [51]. Recall that a process Y is said to be tangent at u to the process Y'_u if:

$$\lim_{r \rightarrow 0} \frac{Y(u + rt) - Y(u)}{r^h} = Y'_u(t), \quad (4)$$

where the limit is understood either in finite dimensional distributions or in the stronger sense of distributions. Note Y'_u may and in general will vary with u .

One approach to defining multistable processes is similar to the one developed for constructing mBm [33]: we consider fields of stochastic processes $X(t, u)$, where t is time and u is an independent parameter that controls the variation of α . We then consider a “diagonal” process $Y(t) = X(t, t)$, which will be, under certain conditions, “tangent” at each point t to a process $t \mapsto X(t, u)$.

A particular class of multistable processes, termed “linear multistable multifractional motions” (lmmm) takes the following form [11], [10]. Let (E, \mathcal{E}, m) be a σ -finite measure space, and Π be a Poisson process on $E \times \mathbb{R}$ with mean measure $m \times \mathcal{L}$ (\mathcal{L} denotes the Lebesgue measure). An lmmm is defined as:

$$Y(t) = a(t) \sum_{(X,Y) \in \Pi} \mathbf{Y}^{<-1/\alpha(t)>} \left(|t - X|^{h(t)-1/\alpha(t)} - |X|^{h(t)-1/\alpha(t)} \right) \quad (t \in \mathbb{R}). \quad (5)$$

where $x^{\langle y \rangle} := \text{sign}(x)|x|^y$, $a : \mathbb{R} \rightarrow \mathbb{R}^+$ is a C^1 function and $\alpha : \mathbb{R} \rightarrow (0, 2)$ and $h : \mathbb{R} \rightarrow (0, 1)$ are C^2 functions.

In fact, lmmm are somewhat more general than said above: indeed, the couple (h, α) allows to prescribe at each point, under certain conditions, both the pointwise Hölder exponent and the local intensity of jumps. In this sense, they generalize both the mBm and the linear multifractional stable motion [65]. From a broader perspective, such multistable multifractional processes are expected to provide relevant models for TCP traces, financial logs, EEG and other phenomena displaying time-varying regularity both in terms of Hölder exponents and discontinuity structure.

Figure 2 displays a graph of an lmmm with linearly increasing α and linearly decreasing H . One sees that the path has large jumps at the beginning, and almost no jumps at the end. Conversely, it is smooth (between jumps) at the beginning, but becomes jaggier and jaggier as time evolves.

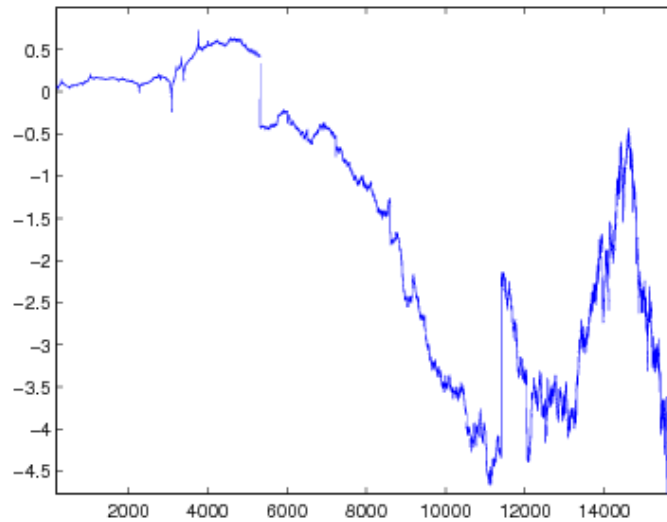


Figure 2. Linear multistable multifractional motion with linearly increasing α and linearly decreasing H

Multiparameter processes

In order to use stochastic processes to represent the variability of multidimensional phenomena, it is necessary to define extensions for indices in \mathbb{R}^N ($N \geq 2$) (see [58] for an introduction to the theory of multiparameter processes). Two different kinds of extensions of multifractional Brownian motion have already been considered: an isotropic extension using the Euclidean norm of \mathbb{R}^N and a tensor product of one-dimensional processes on each axis. We refer to [16] for a comprehensive survey.

These works have highlighted the difficulty of giving satisfactory definitions for increment stationarity, Hölder continuity and covariance structure which are not closely dependent on the structure of \mathbb{R}^N . For example, the Euclidean structure can be unadapted to represent natural phenomena.

A promising improvement in the definition of multiparameter extensions is the concept of *set-indexed processes*. A set-indexed process is a process whose indices are no longer “times” or “locations” but may be some compact connected subsets of a metric measure space. In the simplest case, this framework is a

generalization of the classical multiparameter processes [54]: usual multiparameter processes are set-indexed processes where the indexing subsets are simply the rectangles $[0, t]$, with $t \in \mathbb{R}_+^N$.

Set-indexed processes allow for greater flexibility, and should in particular be useful for the modeling of censored data. This situation occurs frequently in biology and medicine, since, for instance, data may not be constantly monitored. Censored data also appear in natural terrain modeling when data are acquired from sensors in presence of hidden areas. In these contexts, set-indexed models should constitute a relevant frame.

A set-indexed extension of fBm is the first step toward the modeling of irregular phenomena within this more general frame. In [21], the so-called *set-indexed fractional Brownian motion (sifBm)* was defined as the mean-zero Gaussian process $\{\mathbf{B}_U^H; U \in \mathcal{A}\}$ such that

$$\forall U, V \in \mathcal{A}; \quad E[\mathbf{B}_U^H \mathbf{B}_V^H] = \frac{1}{2} \left[m(U)^{2H} + m(V)^{2H} - m(U \Delta V)^{2H} \right]$$

where \mathcal{A} is a collection of connected compact subsets of a measure metric space and $0 < H \leq \frac{1}{2}$.

This process appears to be the only set-indexed process whose projection on increasing paths is a one-parameter fractional Brownian motion [20]. The construction also provides a way to define fBm's extensions on non-euclidean spaces, *e.g.* indices can belong to the unit hyper-sphere of \mathbb{R}^N . The study of fractal properties needs specific definitions for increment stationarity and self-similarity of set-indexed processes [23]. We have proved that the sifBm is the only Gaussian set-indexed process satisfying these two (extended) properties.

In the specific case of the indexing collection $\mathcal{A} = \{[0, t], t \in \mathbb{R}_+^N\} \cup \{\emptyset\}$, the sifBm can be seen as a multiparameter extension of fBm which is called *multiparameter fractional Brownian motion (MpfBm)*. This process differs from the Lévy fractional Brownian motion and the fractional Brownian sheet, which are also multiparameter extensions of fBm (but do not derive from set-indexed processes). The local behaviour of the sample paths of the MpfBm has been studied in [14]. The self-similarity index H is proved to be the almost sure value of the local Hölder exponent at any point, and the Hausdorff dimension of the graph is determined in function of H .

The increment stationarity property for set-indexed processes, previously defined in the study of the sifBm, allows to consider set-indexed processes whose increments are independent and stationary. This generalizes the definition of Bass-Pyke and Adler-Feigin for Lévy processes indexed by subsets of \mathbb{R}^N , to a more general indexing collection. We have obtained a Lévy-Khintchine representation for these set-indexed Lévy processes and we also characterized this class of Markov processes.

4. Application Domains

4.1. Application: uncertainties management

Our theoretical works are motivated by and find natural applications to real-world problems in a general frame generally referred to as uncertainty management, that we describe now.

Since a few decades, modeling has gained an increasing part in complex systems design in various fields of industry such as automobile, aeronautics, energy, etc. Industrial design involves several levels of modeling: from behavioural models in preliminary design to finite-elements models aiming at representing sharply physical phenomena. Nowadays, the fundamental challenge of numerical simulation is in designing physical systems while saving the experimentation steps.

As an example, at the early stage of conception in aeronautics, numerical simulation aims at exploring the design parameters space and setting the global variables such that target performances are satisfied. This iterative procedure needs fast multiphysical models. These simplified models are usually calibrated using high-fidelity models or experiments. At each of these levels, modeling requires control of uncertainties due to simplifications of models, numerical errors, data imprecisions, variability of surrounding conditions, etc.

One dilemma in the design by numerical simulation is that many crucial choices are made very early, and thus when uncertainties are maximum, and that these choices have a fundamental impact on the final performances.

Classically, coping with this variability is achieved through *model registration* by experimenting and adding fixed *margins* to the model response. In view of technical and economical performance, it appears judicious to replace these fixed margins by a rigorous analysis and control of risk. This may be achieved through a probabilistic approach to uncertainties, that provides decision criteria adapted to the management of unpredictability inherent to design issues.

From the particular case of aircraft design emerge several general aspects of management of uncertainties in simulation. Probabilistic decision criteria, that translate decision making into mathematical/probabilistic terms, require the following three steps to be considered [50]:

1. build a probabilistic description of the fluctuations of the model's parameters (*Quantification of uncertainty sources*),
2. deduce the implication of these distribution laws on the model's response (*Propagation of uncertainties*),
3. and determine the specific influence of each uncertainty source on the model's response variability (*Sensitivity Analysis*).

The previous analysis now constitutes the framework of a general study of uncertainties. It is used in industrial contexts where uncertainties can be represented by *random variables* (unknown temperature of an external surface, physical quantities of a given material, ... at a given *fixed time*). However, in order for the numerical models to describe with high fidelity a phenomenon, the relevant uncertainties must generally depend on time or space variables. Consequently, one has to tackle the following issues:

- *How to capture the distribution law of time (or space) dependent parameters, without directly accessible data?* The distribution of probability of the continuous time (or space) uncertainty sources must describe the links between variations at neighbor times (or points). The local and global regularity are important parameters of these laws, since it describes how the fluctuations at some time (or point) induce fluctuations at close times (or points). The continuous equations representing the studied phenomena should help *to propose models for the law of the random fields*. Let us notice that interactions between various levels of modeling might also be used to derive distributions of probability at the lowest one.
- The navigation between the various natures of models needs a kind of *metric* which could *mathematically describe the notion of granularity or fineness* of the models. Of course, the local regularity will not be totally absent of this mathematical definition.
- All the various levels of conception, preliminary design or high-fidelity modelling, require *registrations by experimentation* to reduce model errors. This *calibration* issue has been present in this frame since a long time, especially in a deterministic optimization context. The random modeling of uncertainty requires the definition of a systematic approach. The difficulty in this specific context is: statistical estimation with few data and estimation of a function with continuous variables using only discrete setting of values.

Moreover, a multi-physical context must be added to these questions. The complex system design is most often located at the interface between several disciplines. In that case, modeling relies on a coupling between several models for the various phenomena and design becomes a *multidisciplinary optimization* problem. In this uncertainty context, the real challenge turns robust optimization to manage technical and economical risks (risk for non-satisfaction of technical specifications, cost control).

We participate in the uncertainties community through several collaborative research projects (ANR and Pôle SYSTEM@TIC), and also through our involvement in the MASCOT-NUM research group (GDR of CNRS). In addition, we are considering probabilistic models as phenomenological models to cope with uncertainties in the DIGITEO ANIFRAC project. As explained above, we focus on essentially irregular phenomena, for which

irregularity is a relevant quantity to capture the variability (e.g. certain biomedical signals, terrain modeling, financial data, etc.). These will be modeled through stochastic processes with prescribed regularity.

4.2. Design of complex systems

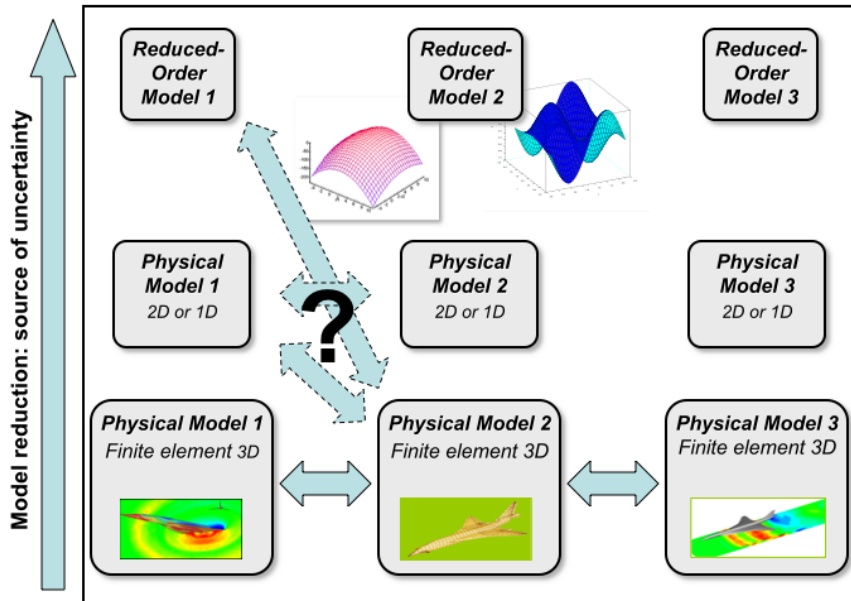


Figure 3. Coupling uncertainty between heterogeneous models

The design of a complex (mechanical) system such as aircraft, automobile or nuclear plant involves numerical simulation of several interacting physical phenomena: CFD and structural dynamics, thermal evolution of a fluid circulation, ... For instance, they can represent the resolution of coupled partial differential equations using finite element method. In the framework of uncertainty treatment, the studied “phenomenological model” is a chaining of different models representing the various involved physical phenomena. As an example, the pressure field on an aircraft wing is the result of both aerodynamic and structural mechanical phenomena. Let us consider the particular case of two models of partial differential equations coupled by limit conditions. The direct propagation of uncertainties is impossible since it requires an exploration and then, many calls to costly models. As a solution, engineers use to build reduced-order models: the complex high-fidelity model is substituted with a CPU less costly model. The uncertainty propagation is then realized through the simplified model, taking into account the approximation error (see [46]).

Interactions between the various models are usually explicitated at the finest level (cf. Fig. 3). How may this coupling be formulated when the fine structures of exchange have disappeared during model reduction? How can be expressed the interactions between models at different levels (in a multi-level modeling)? The ultimate question would be: how to choose the right level of modeling with respect to performance requirements?

In the multi-physical numerical simulation, two kinds of uncertainties then coexist: the uncertainty due to substitution of high-fidelity models with approximated reduced-order models, and the uncertainty due to the new coupling structure between reduced-order models.

According to the previous discussion, the uncertainty treatment in a multi-physical and multi-level modeling implies a large range of issues, for instance numerical resolutions of PDE (which do not enter into the research topics of *Regularity*). Our goal is to contribute to the theoretical arsenal that allows to fly among the different levels of modeling (and then, among the existing numerical simulations). We will focus on the following three axes:

- In the case of a phenomenon represented by two coupled partial differential equations whose resolution is represented by reduced-order models, how to define a probabilistic model of the coupling errors? In connection with our theoretical development, we plan to characterize the regularity of this error in order to quantify its distribution. This research axis is supported by an ANR grant (OPUS project).
- The multi-level modeling assumes the ability to choose the right level of details for the models in adequacy to the goals of the study. In order to do that, a rigorous mathematical definition of the notion of *model fineness/granularity* would be very helpful. Again, a precise analysis of the fine regularity of stochastic models is expected to give elements toward a precise definition of granularity. This research axis is supported by a P  le SYSTEM@TIC grant (EHPOC project), and also by a collaboration with EADS.
- Some fine characteristics of the phenomenological model may be used to define the probabilistic behaviour of its variability. The action of modeling a phenomena can be seen as an interpolation issue between given observations. This interpolation can be driven by physical evolution equations or fine analytical description of the physical quantities. We are convinced that H  lder regularity is an essential parameter in that context, since it captures how variations at a given point induce variations at its neighbors. Stochastic processes with prescribed regularity (see section 3.3) have already been used to represent various fluctuating phenomena: Internet traffic, financial data, ocean floor. We believe that these models should be relevant to describe solutions of PDE perturbed by uncertain (random) coefficients or limit conditions. This research axis is supported by a P  le SYSTEM@TIC grant (CSDL project).

4.3. Biomedical Applications

ECG analysis and modeling

ECG and signals derived from them are an important source of information in the detection of various pathologies, including *e.g.* congestive heart failure, arrhythmia and sleep apnea. The fact that the irregularity of ECG bears some information on the condition of the heart is well documented (see *e.g.* the web resource <http://www.physionet.org>). The regularity parameters that have been studied so far are mainly the box and regularization dimensions, the local H  lder exponent and the multifractal spectrum [53], [55]. These have been found to correlate well with certain pathologies in some situations. From a general point of view, we participate in this research area in two ways.

- First, we use refined regularity characterizations, such as the regularization dimension, 2-microlocal analysis and advanced multifractal spectra for a more precise analysis of ECG data. This requires in particular to test current estimation procedures and to develop new ones.
- Second, we build stochastic processes that mimic in a faithful way some features of the dynamics of ECG. For instance, the local regularity of RR intervals, estimated in a parametric way based on a modeling by an mBm, displays correlations with the amplitude of the signal, a feature that seems to have remained unobserved so far [3]. In other words, RR intervals behave as SRP. We believe that modeling in a simplified way some aspects of the interplay between the sympathetic and parasympathetic systems might lead to an SRP, and to explain both this self-regulating property and the reasons behind the observed multifractality of records. This will open the way to understanding how these properties evolve under abnormal behaviour.

Pharmacodynamics and patient drug compliance

Poor adherence to treatment is a worldwide problem that threatens efficacy of therapy, particularly in the case of chronic diseases. Compliance to pharmacotherapy can range from 5% to 90%. This fact renders clinical tested therapies less effective in ambulatory settings. Increasing the effectiveness of adherence interventions has been placed by the World Health Organization at the top list of the most urgent needs for the health system. A large number of studies have appeared on this new topic in recent years [67], [66]. In collaboration with the pharmacy faculty of Montré@al university, we consider the problem of compliance within the context of multiple dosing. Analysis of multiple dosing drug concentrations, with common deterministic models, is usually based on patient full compliance assumption, *i.e.*, drugs are administered at a fixed dosage. However, the drug concentration-time curve is often influenced by the random drug input generated by patient poor adherence behaviour, inducing erratic therapeutic outcomes. Following work already started in Montré@al [60], [61], we consider stochastic processes induced by taking into account the random drug intake induced by various compliance patterns. Such studies have been made possible by technological progress, such as the “medication event monitoring system”, which allows to obtain data describing the behaviour of patients.

We use different approaches to study this problem: statistical methods where enough data are available, model-based ones in presence of qualitative description of the patient behaviour. In this latter case, piecewise deterministic Markov processes (PDP) seem a promising path. PDP are non-diffusion processes whose evolution follows a deterministic trajectory governed by a flow between random time instants, where it undergoes a jump according to some probability measure [49]. There is a well-developed theory for PDP, which studies stochastic properties such as extended generator, Dynkin formula, long time behaviour. It is easy to cast a simplified model of non-compliance in terms of PDP. This has allowed us already to obtain certain properties of interest of the random concentration of drug [40]. In the simplest case of a Poisson distribution, we have obtained rather precise results that also point to a surprising connection with infinite Bernoulli convolutions [29], [13], [12]. Statistical aspects remain to be investigated in the general case.

5. Software

5.1. FracLab

Participants: Paul BalanÅa, Jacques LÃ©vy VÃ©hel [correspondant].

FracLab was developed for two main purposes:

1. propose a general platform allowing research teams to avoid the need to re-code basic and advanced techniques in the processing of signals based on (local) regularity.
2. provide state of the art algorithms allowing both to disseminate new methods in this area and to compare results on a common basis.

FracLab is a general purpose signal and image processing toolbox based on fractal, multifractal and local regularity methods. FracLab can be approached from two different perspectives:

- (multi-) fractal and local regularity analysis: A large number of procedures allow to compute various quantities associated with 1D or 2D signals, such as dimensions, Holder and 2-microlocal exponents or multifractal spectra.
- Signal/Image processing: Alternatively, one can use FracLab directly to perform many basic tasks in signal processing, including estimation, detection, denoising, modeling, segmentation, classification, and synthesis.

A graphical interface makes FracLab easy to use and intuitive. In addition, various wavelet-related tools are available in FracLab.

FracLab is a free software. It mainly consists of routines developed in MatLab or C-code interfaced with MatLab. It runs under Linux, MacOS and Windows environments. In addition, a “stand-alone” version (*i.e.* which does not require MatLab to run) is available.

FracLab has been downloaded several thousands of times in the last years by users all around the world. A few dozens laboratories seem to use it regularly, with more than two hundreds registered users. Our ambition is to make it the standard in fractal softwares for signal and image processing applications. We have signs that this is starting to become the case. To date, its use has been acknowledged in more than two hundreds research papers in various areas such as astrophysics, chemical engineering, financial modeling, fluid dynamics, internet and road traffic analysis, image and signal processing, geophysics, biomedical applications, computer science, as well as in mathematical studies in analysis and statistics (see <http://fraclab.saclay.inria.fr/> for a partial list with papers). In addition, we have opened the development of FracLab so that other teams worldwide may contribute. Additions have been made by groups in Australia, England, the USA, and Serbia.

6. New Results

6.1. White Noise-based Stochastic Calculus with respect to Multifractional Brownian Motion

Participants: Joachim Lebovits, Jacques Lévy Véhel.

The purpose of this work is to build a stochastic calculus with respect to (mBm) with a view to applications in finance and particularly to stochastic volatility models. We use an approach based on white noise theory.

6.1.1. White Noise-based Stochastic Calculus with respect to multifractional Brownian motion

The following results may be found in [28]. Integration with respect to mBm requires stochastic spaces in which we can differentiate or integrate stochastic processes. Considering the probability space $(\mathcal{S}'(\mathbb{R}), \mathbb{B}(\mathcal{S}'(\mathbb{R})), \mu)$ where μ is probability measure given by BÄ¶chner Minlos theorem, one can build to spaces, noted (\mathcal{S}) and (\mathcal{S}^*) which will play an analogous role to the spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ for tempered distributions. We recall that $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions which are infinitely differentiable and $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions. Let us moreover note (L^2) the space of random variables defined on the probability space $(\mathcal{S}'(\mathbb{R}), \mathbb{B}(\mathcal{S}'(\mathbb{R})), \mu)$ which admit a second order moment. The mBm $B^{(h)}$ has the following Wiener-ItÄ´ chaos decomposition in (L^2) :

$$B^{(h)}(t) = \sum_{k=0}^{+\infty} \langle [0;t], M_{h(t)}(e_k) \rangle_{L^2(\mathbb{R})} = \sum_{k=0}^{+\infty} \left(\int_0^t M_{h(t)}(e_k)(s) ds \right) \langle \cdot, e_k \rangle \quad (6)$$

where $(e_k)_{k \in \mathbb{N}}$ denotes the family of Hermite functions, defined for every integer k in \mathbb{N} , by $e_k(x) := \pi^{-1/4} (2^k k!)^{-1/2} e^{-x^2/2} h_k(x)$ and where $(h_k)_{k \in \mathbb{N}}$ is the family of Hermite polynomial, defined for every integer k in \mathbb{N} , by $h_k(x) := (-1)^k e^{x^2} \frac{d^k}{dx^k} (e^{-x^2})$. Note moreover that M_H is an operator from $\mathcal{S}(\mathbb{R})$ to $L^2(\mathbb{R})$ for every real H in $(0, 1)$ and $\langle \cdot, e_k \rangle$ is a centered random Gaussian variable with variance equal to 1 for all k in \mathbb{N} . We can now define a process, noted $W^{(h)}$, from \mathbb{R} to (\mathcal{S}^*) , which is the derivative of $B^{(h)}$ in sense of (\mathcal{S}^*) by

$$W^{(h)}(t) = \sum_{k=0}^{+\infty} \left[\frac{d}{dt} \left(\int_0^t M_{h(t)}(e_k)(s) ds \right) \right] \langle \cdot, e_k \rangle. \quad (7)$$

Hence we define integral with respect to mBm of any process $\Phi : \mathbb{R} \rightarrow (\mathcal{S}^*)$ as being the element of (\mathcal{S}^*) given by:

$$\int_{\mathbb{R}} \Phi(s, \omega) dB^{(h)}(s) = \int_{\mathbb{R}} \Phi(s) \diamond W^{(h)}(s) ds (\omega), \quad (8)$$

where \diamond denotes the Wick product on (\mathcal{S}^*) . It is then possible to get Itô formulas and Tanaka formula such as

$$\begin{aligned} \int_0^T \frac{\partial f}{\partial x}(t, B^{(h)}(t)) dB^{(h)}(t) &= f(T, B^{(h)}(T)) - f(0, 0) - \int_0^T \frac{\partial f}{\partial t}(t, B^{(h)}(t)) dt \\ &\quad - \frac{1}{2} \int_0^T \left(\frac{d}{dt}[R_h(t, t)] \right) \frac{\partial^2 f}{\partial x^2}(t, B^{(h)}(t)) dt. \end{aligned} \quad (9)$$

for functions with sub exponential growth and where the last equality holds in L^2 .

Once this stochastic calculus with respect to mBm is defined, we can solve differential equations arising in mathematical finance.

6.1.2. Multifractional stochastic volatility

Multifractional stochastic volatility

The results of this part may be found in [6]. We assume that, under the risk-neutral measure, the forward price of a risky asset is the solution of the S.D.E.

$$\begin{cases} dF_t = F_t \sigma_t dW_t, \\ d \ln(\sigma_t) = \theta(\mu - \ln(\sigma_t)) dt + \gamma_h d^\diamond B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0, \theta > 0, \end{cases} \quad (10)$$

where W and W^σ are two standard Brownian motions and B^h is a multifractional Brownian motion independent of W and W^σ with functional parameter h , which is assumed to be continuously differentiable. We assume that W is decomposed into $\rho dW_t^\sigma + \sqrt{1 - \rho^2} dW_t^F$, where W^F is a Brownian motion independent of W^σ . Note that $d^\diamond B_t^h$ denotes differentiation in the sense of white Noise theory. The solution of the volatility process $(\sigma_t)_{t \in [0, T]}$ is

$$\sigma_t \stackrel{a.s.}{=} \exp \left(\ln(\sigma_0) e^{-\theta t} + \mu(1 - e^{-\theta t}) + \gamma_\sigma \int_0^t e^{\theta(s-t)} dW_s^\sigma + \gamma_h e^{-\theta t} I_t(B^h) \right), \quad (11)$$

where $I_t(B^h) \stackrel{a.s.}{=} e^{\theta t} B_t^h - \theta \int_0^t e^{\theta s} B_s^h ds$.

Since the solution the previous S.D.E. is not explicit for $(F_t)_{t \in [0, T]}$ we use preconditioning and then cubature methods in order to get an approximation of it. This model allows to take into account the well-known "smile" effect of volatility, as well as its evolution at various maturities.

6.1.3. Approximation of mBm by fBms

In [18], we establish that a sequence of well-chosen lumped fractional Brownian motions converges in law to a multifractional Brownian motion. This allows to define stochastic integrals with respect to mBm by "transporting" corresponding stochastic integrals with respect to fBm.

6.2. Sample paths properties of the set-indexed Lévy process

Participant: Erick Herbin.

In collaboration with Prof. Ely Merzbach (Bar Ilan University, Israel).

In [24], the class of set-indexed Lévy processes is considered using the stationarity property defined for the set-indexed fractional Brownian motion in [23]. Following Ivanoff-Merzbach's definitions of an indexing collection \mathcal{A} and its extensions $\mathcal{C}_0 = \{U \setminus V; U, V \in \mathcal{A}\}$ and

$$\mathcal{C} = \left\{ U \setminus \bigcup_{1 \leq i \leq n} V_i; n \in \mathbf{N}; U, V_1, \dots, V_n \in \mathcal{A} \right\},$$

a set-indexed process $X = \{X_U; U \in \mathcal{A}\}$ is called a *set-indexed Lévy process* if the following conditions hold

1. $X_{\emptyset'} = 0$ almost surely, where $\emptyset' = \bigcap_{U \in \mathcal{A}} U$.
2. the increments of X are independent: for all pairwise disjoint C_1, \dots, C_n in \mathcal{C} , the random variables $\Delta X_{C_1}, \dots, \Delta X_{C_n}$ are independent.
3. X has m -stationary \mathcal{C}_0 -increments, i.e. for all integer n , all $V \in \mathcal{A}$ and for all increasing sequences $(U_i)_i$ and $(A_i)_i$ in \mathcal{A} , we have

$$[\forall i, m(U_i \setminus V) = m(A_i)] \Rightarrow (\Delta X_{U_1 \setminus V}, \dots, \Delta X_{U_n \setminus V}) \stackrel{(d)}{=} (\Delta X_{A_1}, \dots, \Delta X_{A_n})$$

4. X is continuous in probability.

On the contrary to previous works of Adler and Feigin (1984) on one hand, and Bass and Pyke (1984) on the other hand, the increment stationarity property allows to obtain explicit expressions for the finite-dimensional distributions of a set-indexed Lévy process. From these, we obtained a complete characterization in terms of Markov properties.

The question of continuity is more complex in the set-indexed setting than for real-parameter stochastic processes. For instance, the set-indexed Brownian motion can be not continuous for some indexing collection. We consider a weaker form of continuity, which studies the possibility of point jumps.

The *point mass jump* of a set-indexed function $x : \mathcal{A} \rightarrow \mathbf{R}$ at $t \in \mathcal{T}$ is defined by

$$J_t(x) = \lim_{n \rightarrow \infty} \Delta x_{C_n(t)}, \quad \text{where } C_n(t) = \bigcap_{\substack{C \in \mathcal{C}_n \\ t \in C}} C \quad (12)$$

and for each $n \geq 1$, \mathcal{C}_n denotes the collection of subsets $U \setminus V$ with $U \in \mathcal{A}_n$ (a finite sub-semilattice which generates \mathcal{A} as $n \rightarrow \infty$) and $V \in \mathcal{A}_n(u)$. A set-indexed function $x : \mathcal{A} \rightarrow \mathbf{R}$ is said *pointwise-continuous* if $J_t(x) = 0$, for all $t \in \mathcal{T}$.

Theorem *Let $\{X_U; U \in \mathcal{A}\}$ be a set-indexed Lévy process with Gaussian increments. Then for any $U_{\max} \in \mathcal{A}$ such that $m(U_{\max}) < +\infty$, the sample paths of X are almost surely pointwise-continuous inside U_{\max} , i.e.*

$$P(\forall t \in U_{\max}, J_t(X) = 0) = 1.$$

In the general case, for all $\epsilon > 0$, For all $U \in \mathcal{A}$ with $U \subset U_{\max}$, we define

$$\begin{aligned} N_U(B) &= \# \{t \in U : J_t(X) \in B\}, \\ X_U^B &= \int_B x \cdot N_U(dx), \end{aligned} \quad (13)$$

for all $B \in \mathcal{B}_\epsilon$, the σ -field generated by the opened subsets of $\{x \in \mathbf{R} : |x| > \epsilon\}$. The sample paths of the set-indexed Lévy processes can be derived from the following Lévy-Ito decomposition proved in [24].

Theorem Let (σ, γ, ν) the generating triplet of the SI Lévy process X . Then X can be decomposed as

$$\forall \omega \in \Omega, \forall U \in \mathcal{A}, \quad X_U(\omega) = X_U^{(0)}(\omega) + X_U^{(1)}(\omega),$$

where

1. $X^{(0)} = \{X_U^{(0)}; U \in \mathcal{A}\}$ is a set-indexed Lévy process with Gaussian increments, with generating triplet $(\sigma, \gamma, 0)$,
2. $X^{(1)} = \{X_U^{(1)}; U \in \mathcal{A}\}$ is the set-indexed Lévy process with generating triplet $(0, 0, \sigma)$, defined for some $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 1$ by

$$\forall \omega \in \Omega_1, \quad \forall U \in \mathcal{A},$$

$$X_U^{(1)}(\omega) = \int_{|x|>1} x N_U(dx, \omega) + \lim_{\epsilon \downarrow 0} \int_{\epsilon < |x| \leq 1} x [N_U(dx, \omega) - m(U)] \nu(dx), \quad (14)$$

where N_U is defined in (13) and the last term of (14) converges uniformly in $U \subset U_{\max}$ (for any given $U_{\max} \in \mathcal{A}$) as $\epsilon \downarrow 0$,

3. and the processes $X^{(0)}$ and $X^{(1)}$ are independent.

6.3. Hölder regularity of Set-Indexed processes

Participants: Erick Herbin, Alexandre Richard.

In collaboration with Prof. Ely Merzbach (Bar Ilan University, Israel).

In the set-indexed framework of Ivanoff and Merzbach ([54]), stochastic processes can be indexed not only by \mathbb{R} but by a collection \mathcal{A} of subsets of a measure and metric space (\mathcal{J}, d, m) , with some assumptions on \mathcal{A} . In [25], we introduce and study some assumptions on the metric indexing collection $(\mathcal{A}, d_{\mathcal{A}})$ in order to obtain a Kolmogorov criterion for continuous modifications of SI stochastic processes. Under this assumption, the collection is totally bounded and a set-indexed process with good incremental moments will have a modification whose sample paths are almost surely Hölder continuous, for the distance $d_{\mathcal{A}}$.

Once this condition is established, we investigate the definition of Hölder coefficients for SI processes. From the real-parameter case, the most straightforward are the local (and pointwise) Hölder exponents around $U_0 \in \mathcal{A}$:

$$\tilde{\alpha}_X(U_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{U, V \in B_{d_{\mathcal{A}}}(U_0, \rho)} \frac{|X_U - X_V|}{d_{\mathcal{A}}(U, V)^\alpha} < \infty \right\}.$$

When the processes are Gaussian, a deterministic counterpart to this exponent is defined as it is in the real-parameter framework. For all $U_0 \in \mathcal{A}$, we proved that almost surely, the random and the deterministic exponents are equal. Also, we proved that for the local exponents, this result holds almost surely, uniformly on \mathcal{A} .

Given the particular structure of \mathcal{A} , other coefficients of Hölder regularity were studied on \mathcal{C} :

$$\mathcal{C} = \left\{ A \setminus \bigcup_{k=1}^n B_k : A, B_1, \dots, B_n \in \mathcal{A}, n \in \mathbb{N} \right\}. \quad (15)$$

On specific subclasses \mathcal{C}^l of \mathcal{C} (satisfying $\cup_{l \geq 1} \mathcal{C}^l = \mathcal{C}$), the local (and pointwise) \mathcal{C}^l -Hölder exponents are defined:

$$\tilde{\alpha}_{X, \mathcal{C}^l}(U_0) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{\substack{U \in B_{d_{\mathcal{A}}}(U_0, \rho) \\ V \in \mathcal{B}^l(U_0, \rho)}} \frac{|\Delta X_{U \setminus V}|}{d_{\mathcal{A}}(U, V)^\alpha} < \infty \right\}, \quad (16)$$

and this definition is proved to be independent of l , leading to the definition of $\tilde{\alpha}_{X, \mathcal{C}}(U_0)$. It is compared to $\tilde{\alpha}_X(U_0)$ and related to the Hölder exponent of the process projected on flows (a flow is a continuous increasing path in \mathcal{A}). This last technique permits to show that the pointwise Hölder exponent of the SIfBm is almost surely uniformly equal to H , the Hurst parameter of the SIfBm. This completes some previous results on the multiparameter fractional Brownian motion.

The last exponent which is studied is the exponent of pointwise continuity:

$$\alpha_X^{pc}(t) = \sup \left\{ \alpha : \limsup_{n \rightarrow \infty} \frac{|\Delta X_{C_n(t)}|}{m(C_n(t))^\alpha} < \infty \right\} \quad (17)$$

for all $t \in \mathcal{T}$, where $C_n(t)$ is the smaller set of \mathcal{C}_n containing t . Almost sure results are also obtained in that case. For instance, the coefficient of pointwise continuity of a SI Brownian motion equals $1/2$ *a.s.*

All these results are finally applied to the SIfBm and the SI Ornstein-Ühlenbeck process ([1]).

6.4. Stochastic 2-microlocal analysis

Participants: Erick Herbin, Paul Balança.

Stochastic 2-microlocal analysis has been introduced in [19] to study the local regularity of stochastic processes. If $X = (X_t)_{t \in \mathbf{R}_+}$ is a stochastic process, then for all $t_0 \in \mathbf{R}_+$, a function $s' \mapsto \sigma_{X, t_0}(s')$ called the 2-microlocal frontier is defined to characterize entirely the local regularity of X at t_0 . In particular, for all $s' \in \mathbf{R}$ such that $\sigma_{X, t_0}(s') \in (0, 1)$, it is defined as

$$\sigma_{X, t_0}(s') = \sup \left\{ \sigma : \limsup_{\rho \rightarrow 0} \sup_{u, v \in B(t_0, \rho)} \frac{|X_u - X_v|}{|u - v|^\sigma \rho^{-s'}} < \infty \right\}.$$

The 2-microlocal frontier gives a more complete picture of the regularity than classical pointwise and local Hölder exponents, which are widely used in the literature. Furthermore, it is stable under the action of (pseudo-)differential operators.

[19] mainly focused on Gaussian processes, and in particular obtained a characterization of the regularity for Wiener integrals $X_t = \int_0^t \eta_u dW_u$, with $\eta \in L^2(\mathbf{R})$.

Our main goal was therefore to extend this result to any stochastic integral

$$X_t = \int_0^t H_u dM_u,$$

where M is a local martingale and H an adapted continuous process.

In fact, in [15], we first reduced this problem to the study of local martingales, and we have shown that almost surely for all $t \in \mathbf{R}_+$, the 2-microlocal frontier of a local martingale M , with quadratic variation $\langle M \rangle$, satisfies

$$\forall s' \geq -\alpha_{M,t}; \quad \sigma_{M,t}(s') = \Sigma_{M,t}(s') = \frac{1}{2} \Sigma_{\langle M \rangle,t}(2s'),$$

where for any process X , $\Sigma_{X,t}$ denotes the pseudo 2-microlocal frontier which is characterized as following

$$\forall s' \in \mathbf{R}; \quad \Sigma_{X,t}(s') = \sigma_{X,t}(s') \wedge (s' + p_{X,t}) \wedge 1,$$

where $p_{X,t}$ corresponds to

$$p_{X,t} = \inf \left\{ n \geq 1 : X^{(n)}(t) \text{ exists and } X^{(n)}(t) \neq 0 \right\},$$

with the usual convention $\inf \{\emptyset\} = +\infty$.

As the previous result is based on Dubins-Schwarz representation theorem, it can be easily extended to characterize the regularity of time-changed multifractional Brownian motions. In this case, we obtain a similar equation where $\frac{1}{2}$ is replaced by $H(t)$, the value of the Hurst function at t .

Using this last equality, we can obtain the regularity of the stochastic integral X previously defined: almost surely for all $t \in \mathbf{R}_+$

$$\forall s' \geq -\alpha_{X,t}; \quad \sigma_{X,t}(s') = \Sigma_{X,t}(s') = \frac{1}{2} \Sigma_{\int_0^\bullet H_u^2 d\langle M \rangle_{u,t}}(2s').$$

In the particular case of an integration with respect to a Brownian motion B , the result can be simplified using the stability under differential operators: for almost all $\omega \in \Omega$ and for all $t \in \mathbf{R}_+$, the 2-microlocal frontier satisfies

1. if $H_t(\omega) \neq 0$:

$$\forall s' \in \mathbf{R}; \quad \sigma_{X,t}(s') = \sigma_{B,t}(s') = \left(\frac{1}{2} + s' \right) \wedge \frac{1}{2};$$

2. if $H_t(\omega) = 0$:

$$\forall s' \geq -\alpha_{X,t}; \quad \sigma_{X,t}(s') = \left(\frac{1}{2} + \frac{\Sigma_{H^2,t}(2s')}{2} \right) \wedge \frac{1}{2},$$

unless H is locally equal to zero at t , which induces in that case: $\sigma_{X,t} = +\infty$.

Based on this last characterization, we were able to study the regularity of stochastic diffusions. In particular, we illustrated our purpose with the square of δ -dimensional Bessel processes which verify the following equation

$$Z_t = x + 2 \int_0^t \sqrt{Z_s} d\beta_s + \delta t.$$

6.5. Tempered multistable measures and processes

Participants: Jacques L  vy V  hel, Lining Liu.

This year, we concentrated on the following points:

- Define a new type of multistable processes called tempered multistable processes.
- Study the short time and long time behaviors of tempered multistable processes.
- Compare the multistable Lévy processes defined by finite-dimensional distributions (characteristic functions), Poisson representation and series representation.

The idea of the construction of tempered multistable measure and processes comes from the paper [63]. The interest of such processes is that they may be chosen to have moments of all orders. In addition, they are martingales. This will allow to construct stochastic (partial) differential equation driven by tempered multistable measures, which may be used to describe certain physical phenomena.

The characteristic function of a tempered multistable process $X(t)$ is

$$\mathbb{E}(\exp iyX(t)) = \exp \left\{ \frac{1}{2} \int_0^t \Gamma(-\alpha(x)) \left[\left(1 - \frac{iy}{\theta}\right)^{\alpha(x)} + \left(1 + \frac{iy}{\theta}\right)^{\alpha(x)} - 2 \right] \theta^{\alpha(x)} dx \right\}.$$

We have investigated the long time and short time behaviors this process:

Short time behavior:

Let $\alpha: \mathbb{R} \rightarrow [a, b] \subseteq (0, 2)$ be continuous. Let $u \in \mathbb{R}$ and suppose that as $v \rightarrow u$,

$$|\alpha(u) - \alpha(v)| = o\left(\frac{1}{|\log|u - v||}\right). \quad (18)$$

Then when $h \rightarrow 0$,

$$h^{-1/\alpha(t)}[X(t + hu) - X(t)] \rightarrow Y_{\alpha(t)}(u) \quad (19)$$

in finite-dimensional-distributions, where

$$Y_{\alpha(t)}(u) = \int \mathbf{1}_{[0, u]}(z) dM_{\alpha(t)}(z),$$

and $M_{\alpha(t)}$ is an $\alpha(t)$ stable measure. In an other word, $X(t) = M[0, t]$ is $1/\alpha(t)$ -localisable at t with local form $Y_{\alpha(t)}$.

Long time behavior:

Let $\alpha: \mathbb{R} \rightarrow [a, b] \subseteq (0, 2)$ be continuous and $\lim_{s \rightarrow \infty} \alpha(s) \rightarrow \alpha$. Then for $h \rightarrow \infty$

$$h^{-1/2}[X(t + hu) - X(t)] \rightarrow \Gamma(2 - \alpha)B(u) \quad (20)$$

in finite-dimensional-distributions, where B is standard Brownian motion.

Let us now describe our work on the multistable L \tilde{A} vy motion. For $0 < a \leq b < 2$ and $\alpha: \mathbb{R} \rightarrow [a, b]$, the multistable Lévy motion M_c defined by finite-dimensional distributions (characteristics function) is the process such that

$$\mathbb{E}(\exp(i \sum_{j=1}^d \theta_j M_c(t_j))) = \exp \left(- \int \left| \sum_{j=1}^d \theta_j \mathbf{1}_{[0, t_j]}(s) \right|^{\alpha(s)} ds \right); \quad (21)$$

There also exist a Poisson representation of multistable L $\tilde{\mathcal{A}}$ vy process M_p :

$$M_p(t) = \sum_{(X,Y) \in \Pi} C_{\alpha(X)} \mathbf{1}_{[0,t]}(X) Y^{<-1/\alpha(X)>}, \quad (22)$$

where (X, Y) be the random point of the Poisson process Π , $t > 0$, $Y^{<-1/\alpha(X)>} = \text{sign}(Y)|Y|^{-1/\alpha(X)}$ and

$$C_{\alpha(X)} = \left(\frac{1}{\Gamma(1 - \alpha(X)) \cos(\frac{\pi}{2}\alpha(X))} \right)^{1/\alpha(X)}; \quad (23)$$

Finally, the series representation of multistable L $\tilde{\mathcal{A}}$ vy motion M_s is

$$M_s(t) = \sum_{i=1}^{\infty} C_{\alpha(U_i)} \gamma_i \Gamma_i^{-1/\alpha(U_i)} \mathbf{1}_{(U_i \leq t)}, \quad (24)$$

where $\{\Gamma_i\}_{i \geq 1}$ is a sequence of arrival times of a Poisson process with unit arrival time, $\{U_i\}_{i \geq 1}$ is a sequence of i.i.d random variables with uniform distribution on $[0, t]$, $\{\gamma_i\}_{i \geq 1}$ is a sequence of i.i.d random variables with distribution $\mathbb{P}(\gamma_i = 1) = \mathbb{P}(\gamma_i = -1) = 1/2$. All three sequences $\{\Gamma_i\}_{i \geq 1}$, $\{U_i\}_{i \geq 1}$ and $\{\gamma_i\}_{i \geq 1}$ are independent, and

$$C_{\alpha(U_i)} = \left(\frac{1}{\Gamma(1 - \alpha(U_i)) \cos(\frac{\pi}{2}\alpha(U_i))} \right)^{1/\alpha(U_i)}. \quad (25)$$

We have proved that these three definitions yield the same process in law.

6.6. Local strings and the CH set

Participant: Jacques L $\tilde{\mathcal{A}}$ vy V $\tilde{\mathcal{A}}$ hel.

In collaboration with Prof. Franklin Mendivil (Acadia University, Canada).

We have extended the definition of fractal strings originally proposed in [59] and modified in [37] to deal with the local behaviour of fractal sets. This allows to analyze the pointwise oscillatory properties of locally self-similar sets ([38]).

We have also analyzed in details the structure of a set build by "stacking" Cantor sets with continuously varying dimensions (see figure 4). The resulting set, called "Christiane's hair" set or CH set, displays a number of interesting properties. Each "strand of hair" is a C^∞ curve. Its Hausdorff dimension is 2. Furthermore, it is Minkowski measurable in dimension 2 with vanishing Minkowski content.

6.7. General models for drug concentration in multi-dosing administration

Participants: Lisandro Fermin, Jacques L $\tilde{\mathcal{A}}$ vy V $\tilde{\mathcal{A}}$ hel.

In collaboration with P.E Lévy Véhel (University of Nice-Sophia-Antipolis and Banque Postale).

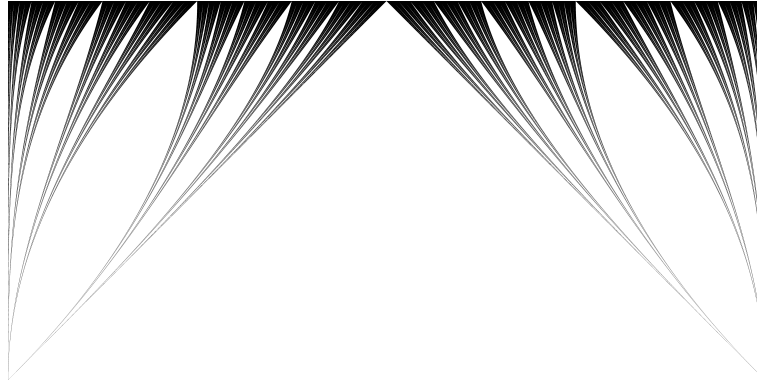


Figure 4. The CH set.

In the past two years, we have developed models for investigating the probability distribution of drug concentration in the case of non-compliance. We have focused on two aspects of practical relevance: the *variability* of the concentration and the *regularity* of its probability distribution. In a first article [29], in a series of three, is considered the case of multi-intravenous dosing using the simplest possible law to model random drug intake, *i.e.* a homogeneous Poisson distribution. In a second article [13], we consider the more realistic multi-oral model, and deal with the complications brought by the first-order kinetics, which are essentially technical. Finally, in [12], we put ourselves in a powerful mathematical frame, known as *Piecewise Deterministic Markov process* (PDMP), that allows us to deal with general drug intake schedules, going beyond the homogeneous Poisson case. We use a PDMP to model the drug concentration in the case of multiple intravenous doses. In this particular model, we consider that the doses administration regimen is modeled by a non-homogeneous Poisson process whose jump rate is controlled by mean of a Markov chain. In this sense our PDMP model is a generalization to the continuous-models studied in [29]. In the following we detail our PDM model and the results obtained in the multi-IV case, see [12].

The model setting

Inspired by the PDMP model given in [47], [48], we consider a drug dosing stochastic regimen defined as follows.

Let us consider $(J_n)_{n \in \mathbb{N}}$ an irreducible Markov chain taking values in the state space $K = \{1, \dots, k\}$ with initial law $\alpha_i = \mathbb{P}(J_0 = i)$ for all $i \in K$ and transition probability matrix $Q = (q_{ij})_{i,j \in K}$. We denote by $(T_n)_{n \in \mathbb{N}}$ the sequence of the random time doses and $(S_n)_{n \in \mathbb{N}}$ the time dose intervals; *i.e.* $S_n = T_{n+1} - T_n$. We consider that the doses administration regimen is modeled by mean of the Markov process $(J_n)_{n \in \mathbb{N}}$ considering the following assumptions:

- The patient takes a dose $D_{J_n} \in \{D_i, i \in K\}$ at the time T_n , where the doses D_i are all different and different of zero.
- The time dose S_n is a random variable with exponential law of parameter $\lambda_{J_n} \in \{\lambda_i, i \in K\}$, where the jump rate λ_i of state i is a positive constant.

We consider that these doses translate into immediate increases of the concentration by the value $d_i = \frac{D_i}{V_d}$ if $J_n = i$, where V_d is the apparent volume of distribution. After that, the effect of the dose taken at time T_n decreases exponentially fast with an exponential rate of elimination k_e .

We define $(\nu_t)_{t \in \mathbb{R}}$ by $\nu_t = \sum_{n \geq 0} J_n \mathbb{1}_{[T_n, T_{n+1}[}(t)$. We denote by $(C_t)_{t \in \mathbb{R}}$ the drug concentration stochastic process which take values on $\mathbb{R}_+^* =]0, \infty[$, we suppose that $\mathbb{P}(C_0 = x) = 1$. Between the jumps, the dynamical evolution of the continuous time process (C_t) is modeled by the flow $\phi(t, x) = x \exp\{-k_e t\}$.

Thus, the sample path of the stochastic process $(C_t)_{t \in \mathbf{R}_+}$ with values in \mathbf{R}_+^* starting from a fixed point x is given by

$$C_t = xe^{-k_e t} + \sum_{i \geq 1} d_{J_i} e^{-k_e(t-T_i)} \mathbb{1}_{(t \geq T_i)}. \quad (26)$$

The process $(C_t, \nu_t)_{t \in \mathbf{R}_+}$ is a PDMP. From [49], we have that the infinitesimal generator \mathcal{U} of $(C_t, \nu_t)_{t \in \mathbf{R}_+}$ is given by

$$\mathcal{U}f(x, i) = -k_e x \frac{d}{dx} f(x, i) + \lambda_i \sum_{j \in K} q_{ij} (f(x + d_j, j) - f(x, i)), \quad (27)$$

with $(x, i) \in E = \mathbf{R}_+^* \times K$ and $f \in \mathbb{D}(\mathcal{U})$ the set of measurable and differentiable on the first argument.

The characteristic function of the concentration

The characteristic function $\varphi_\theta(t, x, i)$ of C_t , given the starting point (x, i) , is the unique solution of the following system

$$\begin{cases} \frac{\partial \varphi_\theta}{\partial t}(t, x, i) = -k_e x \frac{\partial \varphi_\theta}{\partial x}(t, x, i) + \lambda_i \sum_{j \in K} q_{ij} \left(e^{i\theta d_j} e^{-k_e t} \varphi_\theta(t, x, j) - \varphi_\theta(t, x, i) \right), \\ \varphi_\theta(0, x, i) = e^{i\theta x}. \end{cases} \quad (28)$$

Variability of the concentration

From (28) we have that the expectation $m(t, x, i) = \mathbb{E}_{(x, i)}[C_t]$ of C_t , given the starting point (x, i) , is given by

$$m(t, x, i) = xe^{-k_e t} + \sum_{\nu, j \in K} \lambda_\nu q_{\nu j} d_j \int_0^t e^{-k_e(t-s)} P_{i\nu}(s) ds, \quad (29)$$

where $P_{i\nu}(t) = \mathbb{P}(\nu_t = \nu | \nu_0 = i)$. The variance $Var(t, i)$ of C_t , given the initial state i , is given by

$$\begin{aligned} Var(t, i) &= \sum_{\nu, j \in K} \lambda_\nu q_{\nu j} d_j^2 \int_0^t e^{-2k_e(t-s)} P_{i\nu}(s) ds - \left(\sum_{\nu, j \in K} \lambda_\nu q_{\nu j} d_j \int_0^t e^{-k_e(t-s)} P_{i\nu}(s) ds \right)^2 \\ &+ 2 \sum_{\nu, j \in K} \sum_{\nu', j' \in K} \lambda_\nu q_{\nu j} d_j \lambda_{\nu'} q_{\nu' j'} d_{j'} \int_0^t \int_0^{t-s} e^{-k_e(t-s)} P_{i\nu}(s) e^{-k_e(t-s-\tau)} P_{j\nu'}(\tau) d\tau ds. \end{aligned} \quad (30)$$

The distribution of limit concentration

The characteristic function $\varphi(\theta, i)$ of the limit concentration C , given the starting state i , satisfies

$$-k_e \theta \frac{d}{d\theta} \varphi(\theta, i) + \sum_{j \in K} \lambda_j q_{ji} e^{i\theta d_j} \varphi(\theta, j) - \lambda_i \varphi(\theta, i) = 0.$$

Thus, the random variables $C(t)$ converge in distribution, when t tends to infinity, to a well defined random variable C whose characteristic function is

$$\varphi(\theta) = \sum_{j \in K} \varphi(\theta, j).$$

Variability of the limit concentration

We denote by m_i the mean of the limit concentration C in the state $\nu = i$ and $m = \sum_{i \in K} m_i$ the mean of C and Var its variance. Then,

$$\begin{aligned} m &= \frac{1}{k_e} \sum_{i,j \in K} \pi_i \lambda_i q_{ij} d_j. \\ m_i &= \frac{1}{k_e} \sum_{j \in K} \pi_j \lambda_j q_{ji} d_i + \frac{1}{k_e} \left(\sum_{j \in K} \lambda_j q_{ji} m_j - \lambda_i m_i \right). \\ Var &= \frac{1}{2k_e} \sum_{i,j \in K} \pi_i \lambda_i q_{ij} d_j^2 + \frac{1}{k_e} \sum_{i,j \in K} \lambda_i q_{ij} d_j (m_i - \pi_i m). \end{aligned}$$

Regularity of the limit concentration

The characteristic function φ satisfies

$$|\varphi(\theta)| \sim K |\theta|^{-\mu_{max}}, \quad \theta \rightarrow \infty, \quad (31)$$

where K is a positive constant and $\mu_{max} = \max_{i \in K} \frac{\lambda_i}{k_e}$.

This result will allow us to describe in detail aspects of the limit distribution that are important for assessing the efficacy of therapy.

6.8. Complex systems design

Participant: Erick Herbin.

In collaboration with Dassault Aviation, EADS, EDF.

The preliminary design of complex systems can be described as an exploration process of a so-called design space, generated by the global parameters. An interactive exploration, with a decisional visualization goal, needs reduced-order models of the involved physical phenomena. We are convinced that the local regularity of phenomena is a relevant quantity to drive these approximated models. Roughly speaking, in order to be representative, a model needs more informations where the fluctuations are the more important (and consequently, where irregularity is the more important).

In collaboration with Dassault Aviation, EDF and EADS, we study how the local regularity can provide a good quantification of the concept of *granularity* of a model, in order to select the good level of fidelity adapted to a required precision.

Our works in that field can be expressed into:

- The definition and the study of stochastic partial differential equations driven by processes with prescribed regularity (that do not enter into the classical theory of stochastic integration).
- The study of the evolution of the local regularity inside stochastic partial differential equations (SPDE). Stochastic 2-microlocal analysis should provide informations about the local regularity of the solutions, in function of the coefficients of the equations. The knowledge of the fine behaviour of the solution of the SPDE will provide important informations in the view of numerical simulations.

7. Contracts and Grants with Industry

7.1. Grants with Industry

Academic and industrial collaborations are supported by CSDL (Complex Systems Design Lab) project of the Pôle de Compétitivité SYSTEM@TIC PARIS-REGION (11/2009-10/2012). Among the involved industrial partners, we can mention Dassault Aviation, EADS, EDF, MBDA and Renault. The goal of the project is the development of a scientific platform of decisional visualization for preliminary design of complex systems.

8. Partnerships and Cooperations

8.1. Regional Initiatives

The Regularity team collaborates with Supelec (Hana Baili) and with the Department of Mathematics at the University of Nantes (Anne Philippe) in the frame of the DIGITEO ANIFRAC project

8.2. National Initiatives

Regularity participates in the CSDL project of the Pôle de Compétitivité SYSTEM@TIC PARIS-REGION. The academic partners involved are ECP, Ecole des Mines de Paris, ENS Cachan, INRIA, Supelec.

8.3. International Initiatives

8.3.1. INRIA International Partners

- The Regularity team collaborates with Bar Ilan university on theoretical developments around set-indexed fractional Brownian motion and set-indexed Lévy processes (invitations of Erick Herbin in Israël during five months in 2006, 2007, 2008, 2009 and 2011 and invitation of Prof. Ely Merzbach at Ecole Centrale Paris in 2008, 2009, 2010 and 2011). The PhD thesis of Alexandre Richard is supervised in collaboration by Erick Herbin and Ely Merzbach.
- The Regularity team collaborates with Michigan State University (Prof. Yimin Xiao) on the study of fine regularity of multiparameter fractional Brownian motion (invitation of Erick Herbin at East Lansing in 2010).
- The Regularity team collaborates with St Andrews University (Prof. Kenneth Falconer) on the study of multistable processes.
- The Regularity team collaborates with Acadia University (Prof. Franklin Mendivil) on the study of fractal strings.

8.3.2. Visits of International Scientists

Ely Merzbach, from Bar Ilan university (Israel) visited the team for one month. Franklin Mendivil, from Acadia University (Canada), visited the team for one month.

9. Dissemination

9.1. Animation of the scientific community

- Paul Balança attended to the conference *Journées de Probabilités 2011* at Nancy and made a presentation on 2-microlocal analysis, mainly focused on results from [15].
- Alexandre Richard attended to the conference *Journées de Probabilités 2011* at Nancy and made a presentation on Hölder regularity for set indexed-processes, mainly focused on results from [25].
- Joachim Lebovits was invited to give a lecture in the mathematical department of University of Vienna (Austria). He made a presentation at the 35th Stochastic Process and their Applications congress in Oaxaca (Mexico).
- Jacques Lévy Véhel gave an invited lecture at EPFL (Switzerland).
- Erick Herbin was invited to the Israel Mathematical Union 2011 Annual Meeting (Bar-Ilan University, Israel). Talk: "Some recent advances on stochastic 2-microlocal analysis for stochastic processes".
- Erick Herbin was invited to the Geometric Functional Analysis & Probability Seminar (Weizmann Institute of Science, Israel) in July, 2011. Talk: "Several characterisations of the set-indexed Lévy processes".

9.1.1. Organisation committees

Erick Herbin is member of the IMdR Work Group "Uncertainty and industry".

Erick Herbin is member of the CNRS Research Group GDR Mascot Num, devoted to stochastic analysis methods for codes and numerical treatment.

9.1.2. Editorial board

Erick Herbin is reviewer for Mathematical Reviews (AMS).

Jacques Lévy Véhel is associate editor of the journal Fractals.

9.2. Teaching

- Erick Herbin is Director of the Mathematics Department at Ecole Centrale Paris.
- Erick Herbin is in charge of the Probability Course at Ecole Centrale Paris (20h).
- Erick Herbin is in charge of the Random Modeling Course at Ecole Centrale Paris (30h).
- Erick Herbin and Jacques Lévy Véhel are in charge of the Brownian Motion and Stochastic Calculus Course at Ecole Centrale Paris (30h).
- Jacques Lévy Véhel gives a course on wavelets and fractals at Ecole Centrale Nantes (8h).
- Erick Herbin gives travaux dirigés on Real and Complex Analysis at Ecole Centrale Paris (10h).
- Erick Herbin is in charge of the Numerical Simulation Program in the Applied Mathematics option of Ecole Centrale Paris.
- Erick Herbin is supervisor of several student's research projects in the field of Mathematics at Ecole Centrale Paris.
- Paul Balança gives travaux dirigés on Probability (L3) at Ecole Centrale Paris (9h).
- Paul Balança gives travaux dirigés on Real and Complex Analysis (L3) at Ecole Centrale Paris (9h).
- Paul Balança gives travaux dirigés on Random Modeling (M1) at Ecole Centrale Paris (20).
- Joachim Lebovits gives travaux dirigés on Real and Complex Analysis (L3) at Ecole Centrale Paris (9h).

- Joachim Lebovits gives travaux dirigés on Probability (L3) at Ecole Centrale Paris (9h).
- Joachim Lebovits gives travaux dirigés on financial mathematics (M1) at Ecole Centrale Paris (15h).
- Joachim Lebovits gives travaux dirigés on stochastic calculus (M2) at Ecole Centrale Paris (15h).
- Joachim Lebovits supervises students research projects on financial mathematics at Ecole Centrale Paris.
- Alexandre Richard gives travaux dirigés on Probability (L3) at Ecole Centrale Paris (9h).
- Alexandre Richard gives travaux dirigés on Statistics (L3) at Ecole Centrale Paris (9h).
- Alexandre Richard gives travaux dirigés on Random Modeling (M1) at Ecole Centrale Paris (20h).
- Alexandre Richard supervises students research projects on probability at Ecole Centrale Paris (approx. 10h).
- Alexandre Richard supervises students research projects on economic modelling of the cost and efficiency of a technique of hips resurfacing at Ecole Centrale Paris (approx. 15h).
- Benjamin Arras gives travaux dirigés on Probability (L3) at Ecole Centrale Paris (9h).
- Benjamin Arras gives travaux dirigés on Real and Complex Analysis (L3) at Ecole Centrale Paris (9h)
- Benjamin Arras gives travaux dirigés on stochastic calculus (M2) at Ecole Centrale Paris (15h).

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