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Activity Report 2012

Project-Team CONCHA

Complex Flow Simulation Codes based on High-order and Adaptive methods

IN COLLABORATION WITH: Laboratoire de mathématiques et de leurs applications (LMAP)

RESEARCH CENTER
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THEME
Computational models and simulation

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Project-Team CONCHA

Keywords: Adaptive Algorithm, Fluid Dynamics, Finite Elements, Numerical Methods, Object Oriented Programming, Optimization

Creation of the Project-Team: January 01, 2009 .

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2. Overall Objectives

2.1. Objectives

The main objective of this project is the development of innovative algorithms and efficient software tools for the simulation of complex flow problems. Accurate predictions of physical quantities are of great interest in fluid mechanics, for example in order to analyze instabilities, predict forces acting on a body, estimate the flow through an orifice, or predict thermal conductivity coefficients. Due to the complex and highly nonlinear equations to be solved, it is difficult to know in advance how fine the spatial or temporal resolution should be and how detailed a given physical model has to be represented. We propose to develop a systematic approach to these questions based on auto-adaptive methods.

Most of the physical problems under consideration have a three-dimensional character and involve the coupling of models and extremely varying scales. This makes the development of fast numerical methods and efficient implementation a question of feasibility. Our contributions concern modern discretization methods (high-order and adaptivity) and goal-oriented simulation tools (prediction of physical quantities, numerical sensitivities, and parameter identification). Concrete applications originate from aerodynamics, viscoelastic flows, heat transfer, and porous media.

The goal of the **first phase** of the project is to develop flow solvers based on modern numerical methods such as high-order discretization in space and time and self-adaptive algorithms. Adaptivity based on a posteriori error estimators has become a new paradigm in scientific computing, first because of the necessity to give rigorous error bounds, and second because of the possible speed-up of simulation tools. A systematic approach to these questions requires an appropriate variational framework and the development of a posteriori error estimates and adaptive algorithms, as well as sufficiently general software tools able to realize these algorithms. To this end we develop a single common library written in C++ and study at hand of concrete applications the possible benefits and difficulties related to these algorithms in the context of fluid mechanics. The main ingredients of our numerical approach are adaptive finite element discretizations combined with multilevel solvers and hierarchical modeling. We develop different kinds of finite element methods, such as discontinuous (DGFEM) and stabilized finite element methods (SFEM), either based on continuous or non-conforming finite element spaces (NCFEM). The availability of such tools is also a prerequisite for testing advanced physical models, concerning for example turbulence, compressibility effects, and realistic models for viscoelastic flows.

The goal of the **second phase** is to tackle questions going beyond forward numerical simulations: parameter identification, design optimization, and questions related to the interaction between numerical simulations and physical experiments. It appears that many questions in the field of complex flow problems can neither be solved by experiments nor by simulations alone. In order to improve the experiment, the software has to be able to provide information beyond the results of simple simulation. Here, information on sensitivities with respect to selected measurements and parameters is required. The parameters could in practice be as different in nature as a diffusion coefficient and a velocity boundary condition. It is our long-term objective to develop the necessary computational framework and to contribute to the rational interaction between simulation and experiment.

The interdisciplinary collaboration is at the heart of this project. The team consists of mathematicians and physicists, and we develop collaborations with computer scientists.

3. Scientific Foundations

3.1. Challenges related to numerical simulations of complex flows

First, we describe some typical difficulties in our fields of application which require the improvement of established and the development of new methods.

- Coupling of equations and models
  The general equations of fluid dynamics consist in a strongly coupled nonlinear system. Its mathematical nature depends on the precise model, but in general contains hyperbolic, parabolic, and elliptic parts. The spectrum of physical phenomena described by these equations is very large: convection, diffusion, waves... In addition, it is often necessary to couple different models in order to describe different parts of a mechanical system: chemistry, fluid-fluid-interaction, fluid-solid-interaction...
- Robustness with respect to physical parameters
  The values of physical parameters such as diffusion coefficients and constants describing different state equations and material laws lead to different behaviour characterized for example by the Reynolds, Mach, and Weissenberg numbers. Optimized numerical methods are available in many situations, but it remains a challenging problem in some fields of applications to develop robust discretizations and solution algorithms.
- Multiscale phenomena
  The inherent nonlinearities lead to an interplay of a wide range of physical modes, well-known for example from the study of turbulent flows. Since the resolution of all modes is often unreachable, it is a challenging task to develop numerical methods, which are still able to reproduce the essential features of the physical phenomenon under study.

3.2. Stabilized and discontinuous finite element methods

The discontinuous Galerkin method [68], [66], [44], [43] has gained enormous success in CFD due to its flexibility, links with finite volume methods, and its local conservation properties. In particular, it seems to be the most widely used finite element method for the Euler equations [45]. On the other hand, the main drawback of this approach is the large number of unknowns as compared to standard finite element methods. The situation is even worse if one counts the population of the resulting system matrices. In order to find a more efficient approach, it seems therefore important to study the connections with other finite element methods.

In view of the ubiquous problem of large Péclet numbers, stabilization techniques have been introduced since a long time. They are either based on upwinding or additional terms in the discrete variational formulation. The drawback of the first technique is a loss in consistency which generally leads to large numerical diffusion. The grand-father of the second technique is the SUPG/GLS method [57], [67]. Recently, new approaches have been developed, which try do avoid coupling of the different equations due to the residuals. In this context we cite LPS (local projection stabilization) [62], [55], [48][5] and CIP (continuous interior penalty) [58], [59].

3.3. Finite element methods on quadrilateral and hexahedral meshes

The construction of finite element methods on quadrilateral, and particularly, hexahedral meshes can be a complicated task; especially the development of mixed and non-conforming methods is an active field of research. The difficulties arise not only from the fact that adequate degrees of freedom have to be found, but also from the non-constantness of the element Jacobians; an arbitrary hexahedron, which we define as the image of the unit cube under a tri-linear transformation, does in general not have plane faces, which implies for example, that the normal vector is not constant on a side.

In collaboration with Eric Dubach (Associate professor at LMAP) and Jean-Marie Thomas (Former professor at LMAP) we have built a new class of finite element functions (named pseudo-conforming) on quadrilateral and hexahedral meshes. The degrees of freedom are the same as those of classical iso-parametric finite elements but the basis functions are defined as polynomials on each element of the mesh. On general quadrilaterals and hexahedra, our method leads to a non-conforming method; in the particular case of parallelotopes, the new finite elements coincide with the classical ones [61], [60].

3.4. Finite element methods for interface problems

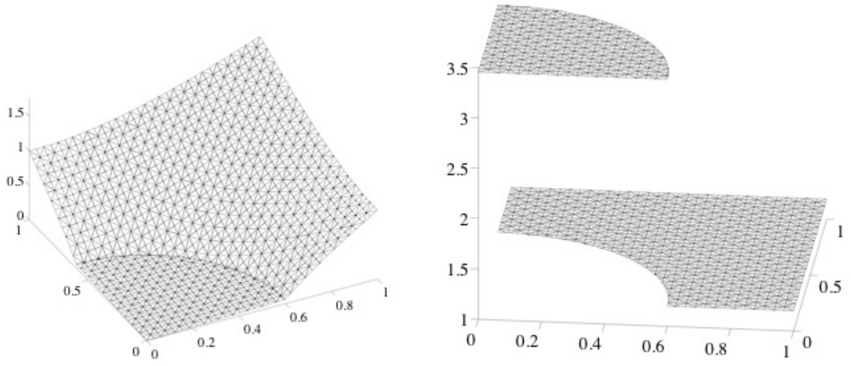


Figure 1. Incompressible elasticity with discontinuous material properties (left: modulus of velocities, right: pressure; from [46]).

The NXFEM (Nitsche eXtended finite element method) has been developed in [63] and [64]. It is based on a pure variational formulation with standard finite element spaces, which are locally enriched in such a way that the accurate capturing of an interface not aligned with the underlying mesh is possible, giving a rigorous formulation of the very popular XFEM. A typical computation for the Stokes problem with varying, piecewise constant viscosity is shown in Figure 1. This technology opens the door to many applications in the field of fluid mechanics, such as immiscible flows, free surface flows and so on.

3.5. Adaptivity

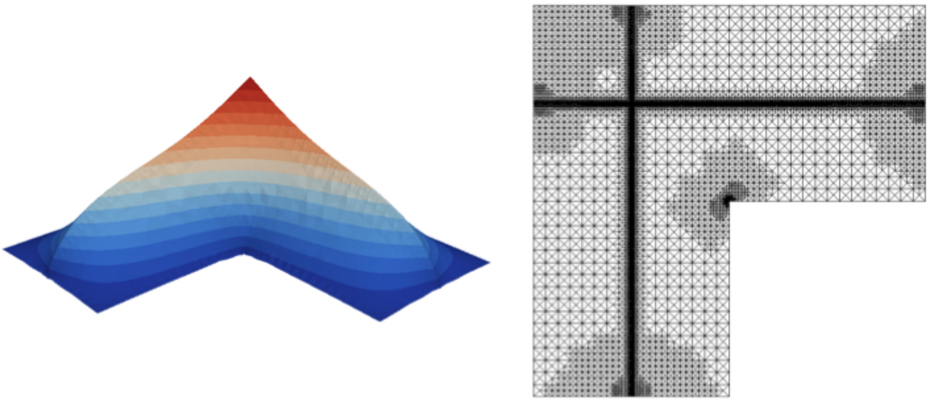


Figure 2. Solution with rough right-hand-side in a corner domain and adaptively refined mesh (from [50]).

Adaptive finite element methods are becoming a standard tool in numerical simulations, and their application in CFD is one of the main topics of Concha. Such methods are based on a posteriori error estimates of the discretization error avoiding explicit knowledge of properties of the solution, in contrast to a priori error estimates. The estimator is used in an adaptive loop by means of a local mesh refinement algorithm. The mathematical theory of these algorithms has for a long time been bounded to the proof of upper and lower bounds, but has made important improvements in recent years. For illustration, a typical sequence of adaptively refined meshes on an L-shaped domain is shown in Figure 2.

The theoretical analysis of mesh-adaptive methods, even in the most standard case of the Poisson problem, is in its infancy. The first important results in this direction concern the convergence of the sequence of solution generated by the algorithm (the standard a priori error analysis does not apply since the global mesh-size does not necessarily go to zero). In order to prove convergence, an unavoidable data approximation term has to be treated in addition to the error estimator [69]. These result do not say anything about the convergence speed, that is the number of unknowns required to achieve a given accuracy. Such complexity estimates are the subject of active research, the first fundamental result in this direction is [54].

Our first contribution [23] to this field has been the introduction of a new adaptive algorithm which makes use of an adaptive marking strategy, which refines according to the data oscillations only if they are by a certain factor larger then the estimator. This algorithm allowed us to prove geometric convergence and quasi-optimal complexity, avoiding additional iteration as used before [71]. We have extended our results to conforming FE without inner node refinement [51] and to mixed FE [50]. In this case, a major additional difficulty arises from the fact that, due to the saddle-point formulation, the orthogonality relation known from continuous FEM does not hold. In addition, we have considered the case of incomplete solution of the discrete systems. To this end, we have developed a simple adaptive stopping criterion based on comparison of the iteration error with the discretization error estimator, see also [49].

Goal-oriented error estimation has been introduced in [52]. It allows to error control and adaptivity directly oriented to the computation of physical quantities, such as the drag and lift coefficient, the Nusselt number, and other physical quantities.

4. Application Domains

4.1. Aerodynamics

Aerodynamics provide a challenging field for numerical simulations in fluid dynamics with a wide range of applications. Robustness of the simulation software with respect to physical parameters as the Reynolds and Mach numbers is necessary condition. In general, realistic simulations need to be done in three dimensions, which makes the efficiency of the numerical approach and implementation a question of feasibility. Therefore, different efforts are made in this project in order to tackle these subjects.

4.2. Red blood cells

Hammou El-Otmany started his PhD thesis in October 2012 in our group, supervised by D. Capatina and D. Graebling. The thesis is financed by UPPA (50%) and CDAPP (50%) and concerns the numerical simulation of biological fluids flows. We will focus more particularly on the physical and numerical modeling of red blood cells.

Clinically, some pathologies such as drepanocytosis or sickle cell anemia are due to the abnormal form of red blood cells. In the microcirculation, where cells must deform to pass through narrow capillaries, the deformability of individual red blood cells is a major determinant of resistance to flow.

The goal is twofold. On the one hand, we want to propose a realistic modeling of red blood cells in artery flow, by taking into account the membrane's viscoelasticity and thus, its deformability. The latter is essentially linked to its structure (i.e. its cellular geometry, membrane properties and cytoplasmic viscosity); thus structural abnormalities, as found in some haematological disorders can be expected to affect blood flow in the microcirculation and/or red cell lifespan.

On the other hand, we want to develop an efficient and stable numerical method in order to treat the coupling between the different models involved: Navier-Stokes for the matrix (blood) and for the cytoplasme (interior of the cell) and a non-Newtonian fluid (for instance, Giesekus) for the membrane. We will use the NXFEM method to take into account the interfaces between fluids.

4.3. Heat transfer

Heat transfer problems involve the coupling of the flow field of the fluid with temperature inside the flow and possibly on the boundary of the flow domain. A typical example of a heat transfer problem is the cooling of a combustion engine, see the project Optimal described in Section 7.1.

4.4. Turbulence

Turbulent flows are ubiquitous in industrial applications. Direct numerical simulation (DNS), which aims at complete resolution of the flow field up to the Kolmogorov scale, has historically been limited to very simple geometries. The increase of computational power and the development of specialized numerical methods open the door to a wider range of applications. However, for most applications of practical interest, the use of some kind of turbulence modeling is unavoidable in order to obtain the prediction of averaged values and commercial software is in general based on such approaches combined with wall laws. In many applications, such as the project Optimal, see Section 7.1, the Reynolds number is at an intermediate level, which means that the turbulence is not fully developed, and the heuristics behind most turbulence models are questionable. Especially, in heat transfer problems, the usage of wall laws seems to considerably lower the accuracy of the predicted mean values. In order to improve the computation of such values, we are particularly interested in variational multiscale methods and its relations to stabilized finite element methods.

4.5. Flows in porous media

Flows in fractured pourous media are very important in petroleum engineering. They represent a good framework for the application of the tools developed in the CONCHA library such as the NXFEM method, goal oriented adaptivity, multiscale coupling of different models and multilevel solvers.

5. Software

5.1. C++ library Concha

Participants: Roland Becker, Daniela Capatina, Robert Luce, David Trujillo.

The objectives of our library CONCHA are to offer a flexible and extensible software with respect to:

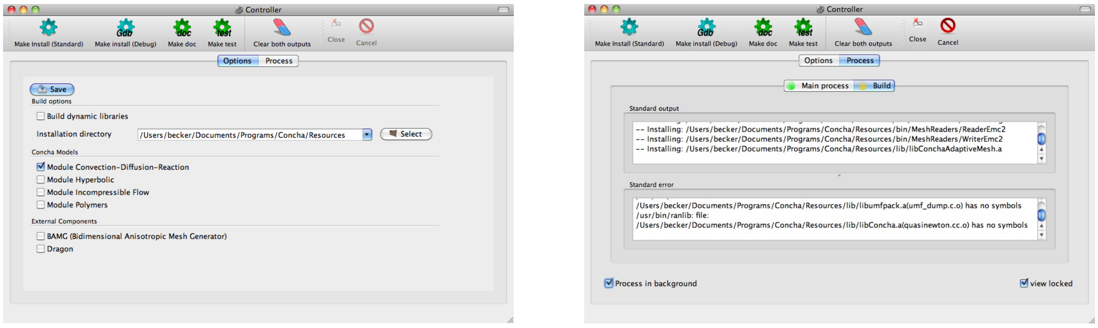
- Numerical methods and
- Physical models.

The aim is to have a flexible code which could easily switch between the different discretizations, in order to provide a toolbox for rapid testing of new ideas.

The software architecture is designed in such a way that a group of core developers can contribute in an efficient manner, and that independent development of different physical applications is possible. Further, in order to accelerate the integration of new members and in order to provide a basis for our educational purposes (see Section 8.1), the software proposes different entrance levels. The basic structure consists of a common block, and several special libraries which correspond to the different fields of applications described in Sections 4.1–4.4 Hyperbolic solvers, Low-Mach number flow solvers, DNS, and viscoelastic flows. A more detailed description of each special library may be found below. In order to coordinate the cooperative development of the library, Concha is based on the Inria-Gforge.

5.2. User interface and python interface

Participants: Roland Becker, David Trujillo.



Build options

☐ Build dynamic libraries

Installation directory /Users/becker/Documents/Programs/Concha/Resources

Concha Models

☑ Module Convection-Diffusion-Reaction
☐ Module Hyperbolic
☐ Module Incompressible Flow
☐ Module Polymers

External Components

☐ BAMG (Bidimensional Anisotropic Mesh Generator)
☐ Dragon

Standard output

```
-- Installing: /Users/becker/Documents/Programs/Concha/Resources/bin/MeshReaders/ReaderEmc2
-- Installing: /Users/becker/Documents/Programs/Concha/Resources/bin/MeshReaders/WriterEmc2
-- Installing: /Users/becker/Documents/Programs/Concha/Resources/lib/libConchaAdaptiveMesh.a
```

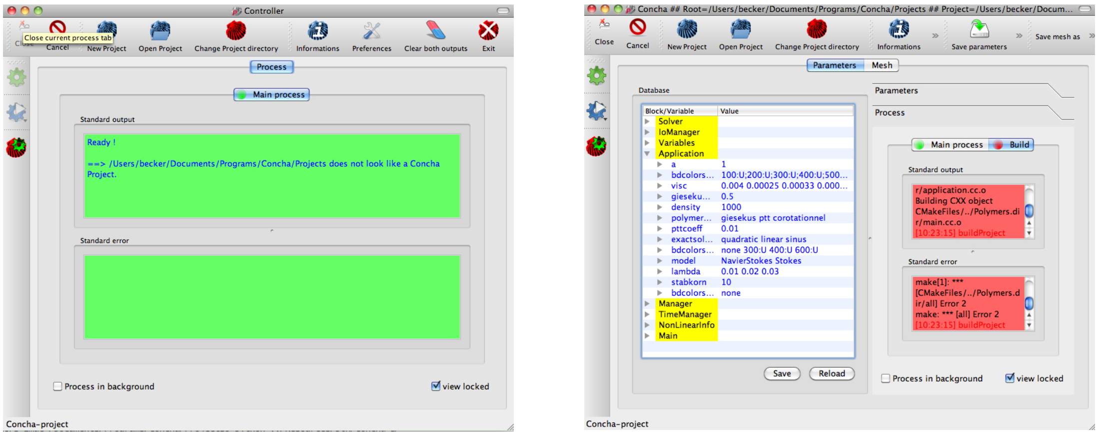
Standard error

```
/Users/becker/Documents/Programs/Concha/Resources/lib/libumfpack.a(umf_dump.c.o) has no symbols
/usr/bin/ranlib: file:
/Users/becker/Documents/Programs/Concha/Resources/lib/libConcha.a(quasinewton.cc.o) has no symbols
```

☑ Process in background ☑ view locked

Figure 3. Graphical user interface: option panel (left) and process panel (right) of the install tool.

We are confronted with heterogenous backgrounds and levels of implication of the developers and users. It seems therefore crucial to be able to respond to the different needs. Our aim is to facilitate the development of the library, and at the same time, to make it possible that our colleagues involved in physical modeling can have access to the functionality of the software with a reasonable investment of time. Two graphical user interfaces have been developed: one for the installation of the library and another one for the building and execution of projects. They are based on common database and scripts written in python. The scripts can also be launched in a shell. In Figure 3 the user interface of the install tool is shown. The option panel allows to choose the components for conditional compilation and the compilation type (debug and release).



```
Ready !

==> /Users/becker/Documents/Programs/Concha/Projects does not look like a Concha
Project.
```

Block/Variable	Value
Solver	
IoManager	
Variables	
Application	
a	1
bdcolors…	100:U;200:U;300:U;400:U;500…
visc	0.004 0.00025 0.00033 0.000…
gieseku…	0.5
density	1000
polymer…	giesekus ptt corotationnel
pttcoeff	0.01
exactsol…	quadratic linear sinus
bdcolors…	none 300:U 400:U 600:U
model	NavierStokes Stokes
lambda	0.01 0.02 0.03
stabkorn	10
bdcolors…	none
Manager	
TimeManager	
NonLinearInfo	
Main	

```
r/application.cc.o
Building CXX object
CMakeFiles/../Polymers.di
r/main.cc.o
[10:23:15] buildProject
```

```
make[1]: ***
[CMakeFiles/../Polymers.d
ir/all] Error 2
make: *** [all] Error 2
[10:23:15] buildProject
```

Figure 4. Graphical user interface: project build panel (left) and parameter panel (right) of the project tool.

In Figure 4 the user interface of the project tool is shown. A project consists of a number of sources files and a parameter file used by the C++-executable. The sources define classes derived from the library, which are used to specify certain data such as boundary conditions and employed finite element spaces. The parameter file contains algorithmic information and physical parameters. It is generated from a database by the python utilities.

The tools offered by this development platform are based on a python interface for the library, called pyConcha. It offers a common interface, based on a pluggin-system, which allows the devloppement of command line tools in parallel. This year the consolidation of the interface part of pyConcha has been an important task. The pyConcha library is now a framework rather than a simple interface to Concha C++ library. It allows now creation of plugins, so that each user-programmer can customize pyConcha to his own goals. Previously, two main programs where working: concha-install.py to install library, and concha-project.py for (semi-)end-users. Both are now plugins of pyConcha, and can be launched by pyConcha at startup. A plugin visualization could now be developped in an independant way, and launched by pyConcha on demand.

The structure of pyConcha framework is clearly splitted in various modules(layers): Command Line Interface module, Graphical User Interface module and Handlers modules, see Figure 5. A great effort has been made for internationalization of pyConcha.

5.3. Euler equations

Participants: Roland Becker, Kossivi Gokpi, Robert Luce, Eric Schall, David Trujillo.

Based on the library CONCHA we have developed a solver for hyperbolic PDE's based on DGFEM. So far different standard solvers for the Euler equations such as Lax-Friedrichs, Steger-Warming, and HLL have been implemented for test problems. A typical example is the scram jet test case shown in Figure 6.

5.4. Incompressible flow solvers

Participants: Roland Becker, Daniela Capatina, Robert Luce, David Trujillo.

We have started the validation of the implementation of different finite element methods for incompressible flows at hand of standard benchmark problems as the Stokes flow around a symmetric cylinder [65] and the stationary flow around a slightly non symmetric cylinder [70], see Figure 7.

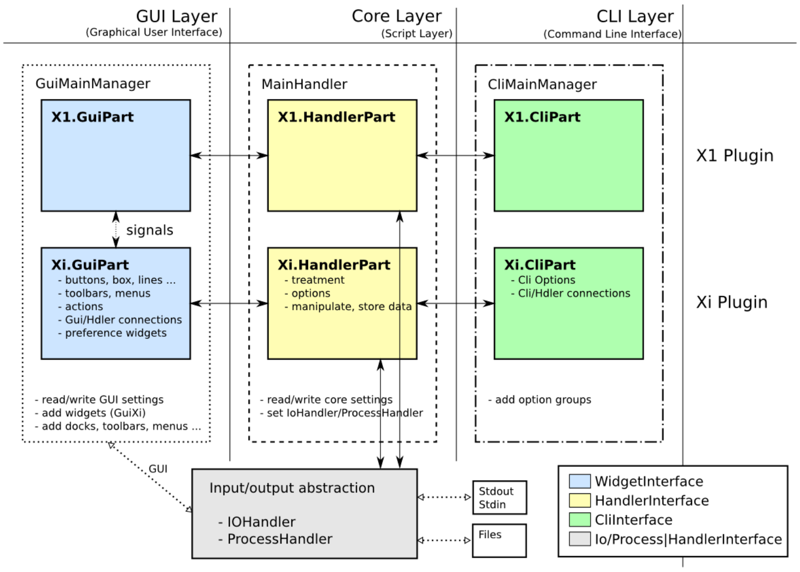


Figure 5. Structure of the pyConcha framework.

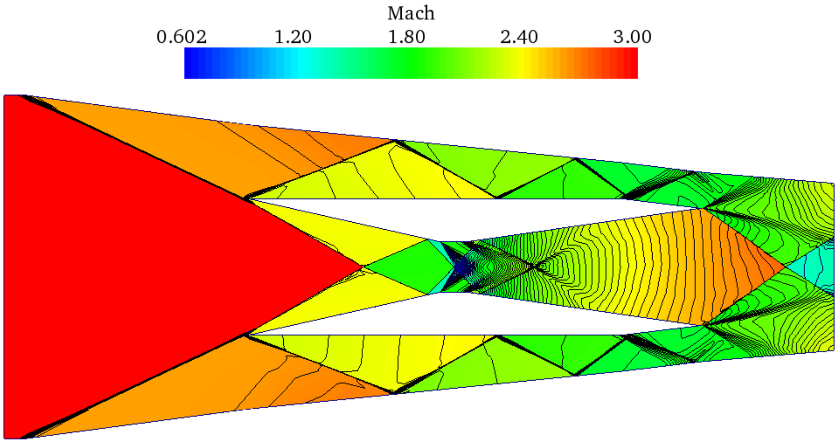


Figure 6. Computed Mach-number distribution for the Scramjet test problem.

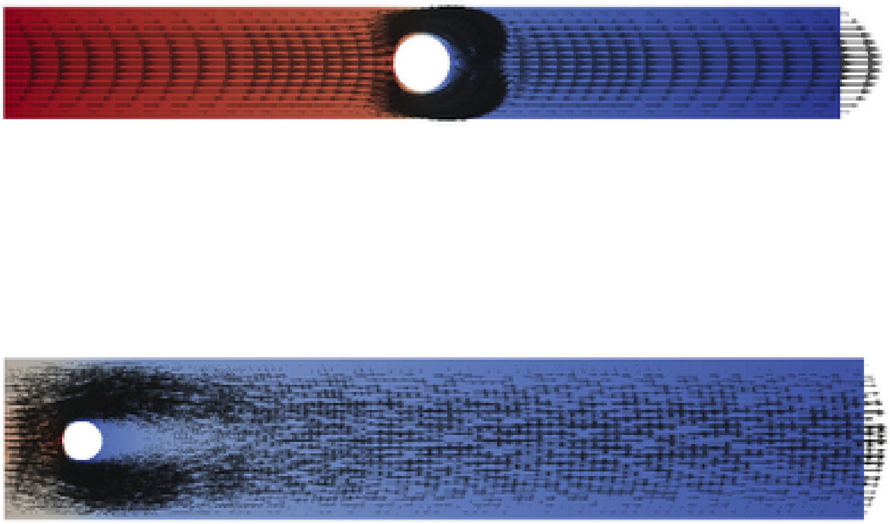


Figure 7. Flow fields for the Stokes (above) and Navier-Stokes (below) benchmark.

5.5. DNS

Participants: Roland Becker, David Trujillo.

For the direct numerical simulation of incompressible turbulent flows, we have started to develop a special solver based on structured meshes with a fast multigrid algorithm incorporating projection-like schemes. The main idea is to use non-conforming finite elements for the velocities with piecewise constant pressures, leading to a special structure of the discrete Schur complement, when an explicit treatment of the convection and diffusion term is used.

5.6. Validation and comparison with other CFD-software

Participants: Roland Becker, Didier Graebling, Eric Schall, David Trujillo.

Validation and comparison with other CFD-software is crucial in order to evaluate the potential of our numerical schemes concerning accuracy, computing time and other practical aspects.

We have compared the Concha library for incompressible and compressible flows. For incompressible flows, we have used a test case proposed by Hulsen and the well-known Schafer-Turek cylinder benchmark in order to validate the accuracy of the Stokes and Navier-Stokes solvers. The viscoelastic code has been compared with PolyFlow for different test configurations.

The compressible Euler code has been compared to the ELSA software developed by ONERA.

For further comparison and validation, it would be important to consider other commercial and research tools such as: *Aéro3* (Inria-Smash), AVBP (CERFACS), Fluent (ANSYS), and OpenFOAM (OpenCfd).

For this purpose we have proposed the ADT-project VALSE in collaboration with a small company involved in aerodynamics (EPSILON Toulouse), which unfortunately has been rejected by Inria.

6. New Results

6.1. Convergence of adaptive finite element algorithms

Participants: Roland Becker, Shipeng Mao, David Trujillo.

The theoretical analysis of mesh-adaptive methods is a very active field of research. We have generalized our previous results concerning optimality of adaptive methods to nonconforming finite elements [53]. Our results include the error due to iterative solution of the system matrices by means of a simple stopping criterion related to the error estimator. The main difficulty was the treatment of the nonconformity which leads to a perturbation of the orthogonality relation at the heart of the proofs for conforming finite elements. We have been able to extend this result to the Stokes equations, considering different lowest-order nonconforming finite elements on triangular and quadrilateral meshes [16].

In [19] we have shown that the smallness assumption required in all former proofs of optimality of adaptive finite element methods can be overcome, at least in some situations.

Finally, we have shown optimality of a new goal-oriented method in [21].

Our theoretical studies, which are motivated by the aim to develop better adaptive algorithms, have been accompanied by software implementation with the Concha library, see Section 5.1. It hopefully opens the door to further theoretical and experimental studies.

6.2. Finite element methods for interface problems

Participants: Nelly Barrau, Roland Becker, Robert Luce.

The original formulation of NXFEM [63] is based on the doubling of elements. In some situations, as the case of a moving interface, it is computationally more convenient to have a method with local enrichment, as for the standard XFEM. In [47] we have developed such an approach based on NXFEM. We have developed an hierarchical formulation for a fictitious domain formulation in [7].

One of the technical difficulties is the simultaneous robustness of the method with respect to the size of the intersection of a mesh cell with the interface and with respect to the discontinuous diffusion parameters. In [ ] (note CRAS 2012) we proposed a modified formulation of the NXFEM which allows us to obtain this robustness to solve the Darcy equation.

In connection with the thesis of Nelly Barrau, supervised by Robert Luce and Eric Dubach (LMAP) we have:

- implemented lots of geometrical tools in 2D and 3D necessary to use the NXFEM methods,
- extended the method to P_k and Q_k finite elements ([42],
- generalized the residual estimator and developed an adaptative process with hanging node (8),
- adapted the method to the transport equation.

6.3. A posteriori error estimators based on $H(\mathrm{div})$-reconstructed fluxes

Participants: Roland Becker, Daniela Capatina, Robert Luce.

Mesh adaptivity is nowadays an essential tool in numerical simulations; in order to achieve it, reliable and efficient, easily computable *a posteriori* error estimators are needed. Such estimators obtained by reconstructing locally conservative fluxes in the Raviart-Thomas finite element space have been largely employed in the past years.

We have so far considered the convection-diffusion equation and proposed a unified framework for several finite element approximations (conforming, nonconforming and discontinuous Galerkin). The main advantage of our approach is to use, contrarily to the existing references, only the primal mesh for the flux reconstruction, which presents certain facilities from a computational point of view.

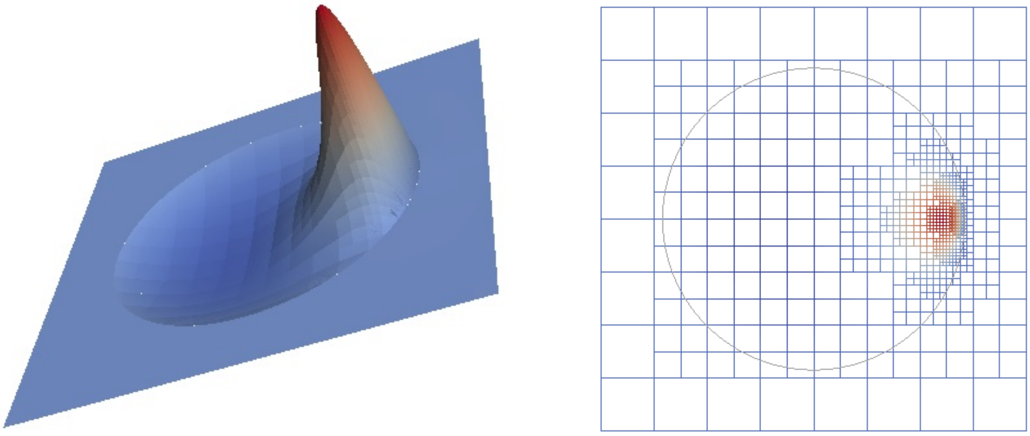


Figure 8. Result of an adaptative process with hanging node

For this purpose, the construction of the $H(\mathrm{div})$-vector involved in the error estimator is inspired by the hypercircle method cf. [56] and is achieved on patches, which may overlap. A patch depends on the type of the employed finite elements and is defined as the support of a basis function.

Our first results were presented in [12]. We are working on the extension to higher-order approximations, to quadrilateral meshes and to other model problems.

6.4. Discretization of Euler's equations

Participants: Roland Becker, Kossivi Gopki, Eric Schall, David Trujillo.

Over the past years, significant advances have been made in developing discontinuous Galerkin finite element methods (DGFEM) for applications in fluid flow and heat transfer. Certain features of the method have made it attractive as an alternative to other popular methods such as finite volume and more convenient finite element methods in thermal fluid engineering analyses. The DGFEM has been used successfully to solve hyperbolic systems of conservation laws. It makes use of the same local function space as the continuous method, but with relaxed continuity at inter-element boundaries. Since it uses discontinuous piecewise polynomial bases, the discretization is locally conservative and in the considered lowest-order case, the method preserves the maximum principle for scalar equations.

One of the challenges in Computational Fluid Dynamic (CFD) is to obtain as accurate as possible the solution of the problem under consideration at very low cost in terms of computational time. So our principal work is to find some relevant and robust strategies and technics of meshes adaptation in order to concentrate just the calculation where there are physical phenomena to capture. From Industrial point of view, the aim is to get the stationary solution as quick as possible with as much accuracy as possible. The main limitation of these results in CFD concern the underlying models: for example, nearly nothing seems to be known for (even linear) first-order systems or for realistic nonlinear equations. We therefore have developed different modern techniques, especially adaptive methods, to tackle this kind of problems in compressible CFD. The strategy is to iteratively improve the quality of the approximate solutions based on computed information (a posteriori error analysis). In this way, a sequence of locally refined meshes is constructed, which allows for better efficiency as compared to more classical approaches in the presence of different kind of singularities. The main goal is to improve the aerodynamical design process for complex configurations by significantly reducing the time from geometry to solution at engineering-required accuracy using high-order adaptive methods.

One of our strategies of refinement is based on the creation of hanging nodes commonly called non-conforming refinement. The figures 9 show superposition of two kinds of meshes. One is a non-conforming refined mesh (black color) and the other one is the initial grid (red color) on which the refinement has been performed. It shows the technic of cutting the cells where singularities occur in the scramjet inlet.

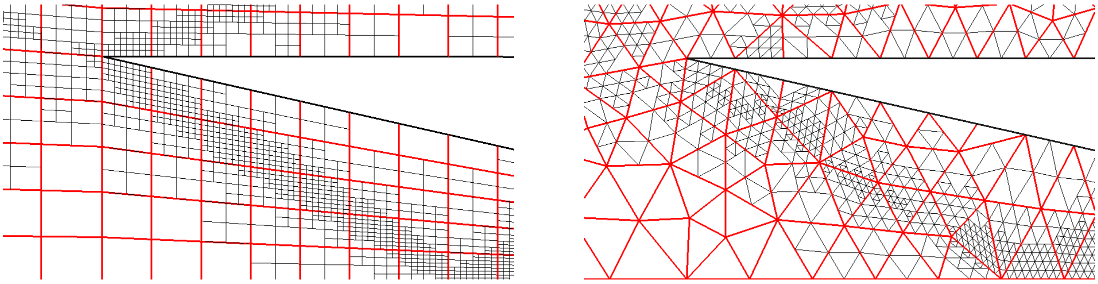

(a) (b)

Figure 9. Superposition of non-conforming adapted black color) grid and initial grid (red color) – (a) quadrangles and (b) triangles.

The mesh adaptation is designed using some criteria as a posteriori error estimates. We have designed criteria based on the calculation of the jump of physical quantities like density, pressure, entropy, temperature and mach number at the inter-element. This criteria seems to be a very good indicator for the mesh adaptation. Figure 10 is the comparison of isoline of the density in scramjet internal flow at mach 3 of the initial mesh, the third and the sixth mesh after refinement.The indicator used is the density jump. It shows the impact and the accuracy of the solution obtained after the sixth iteration of the refinement.

The figure 11 shows the streamlines of the density in the scramjet inlet after the seventh iteration. This shows how the adaptation depicts almost clearly and accurately the shock waves and the expansion waves and their interactions in the domain.

Figure 12 represent the density isolines of a flow past cylinder test case using the non-conforming mesh adaptation with quadrangular an triangular girds.

We have also settled another indication which is hierarchical. It measures the difference of g_h with the physical quantity $g_{h/2}$ obtained by computation on a globally refined mesh $h/2$. This allows us the make comparison with the previous indicator. The case test considered for this comparison is an external flows past a cylinder airfoil at fixed free stream conditions : $M_\infty = 3$. The result is quite surprising the way one type of indicator can capture phenomenon that are not capture by the another one. In fact the hierarchical indicator seems to capture recirculation downstream to the obstacle which was not capture by the jump indicator (see figure 13)

We compare the computational time between a non-conforming mesh refinement and a globally mesh refined with nearly the same amount of cells. The meshes contain quadrangles or triangles. We can observe trough the following tables that the adapted meshes wether triangular or quadrangular meshes allow to save 20 to 90 times the computational time than the normal globally refined mesh. (see tables 1 and 2)

In table 1, the gain in time is 35 times in quadrangular grid case and 90 times triangular ones and in table 2, the gain in time: 18 times in quadrangular grid case and 58 times triangular ones. So one can say that the adaptive mesh with the strategies and technics we have settled are efficient and robust in capturing physical phenomenon at a very reasonable low cost.

In concluding, the procedure of refinement permit to save computational time and have good accuracy of the approximated solution computed. Our focus is to continue the improve our methods and strategies in order to meet the requirement of accuracy, robustness and efficiency. Many other works are in hand such as

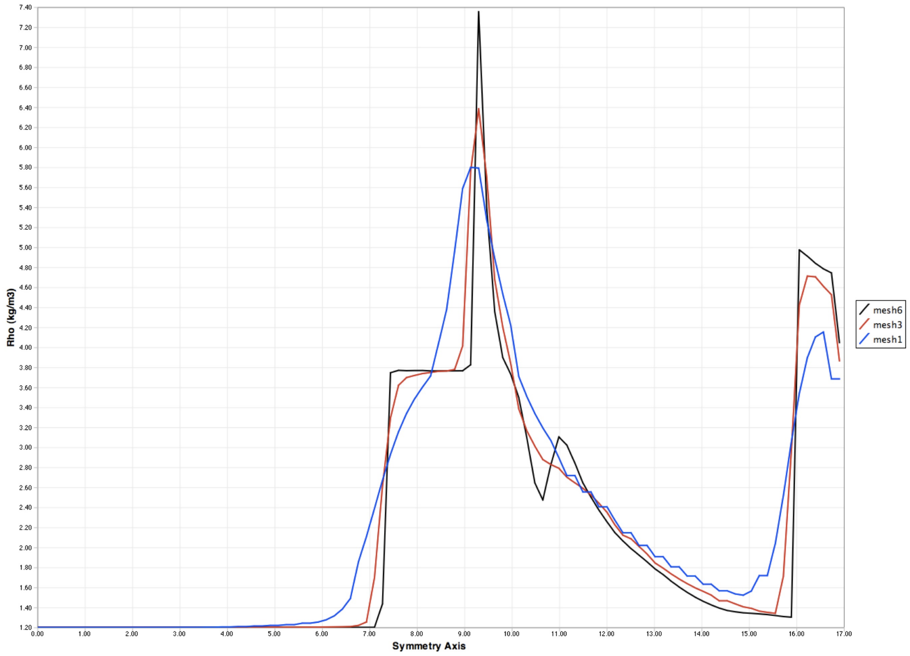


Figure 10. Cutlines along the symmetry axis of various meshes for the scramjet test case

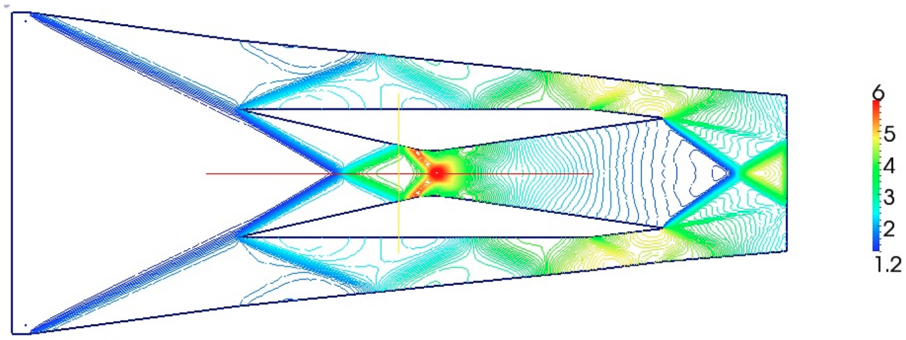
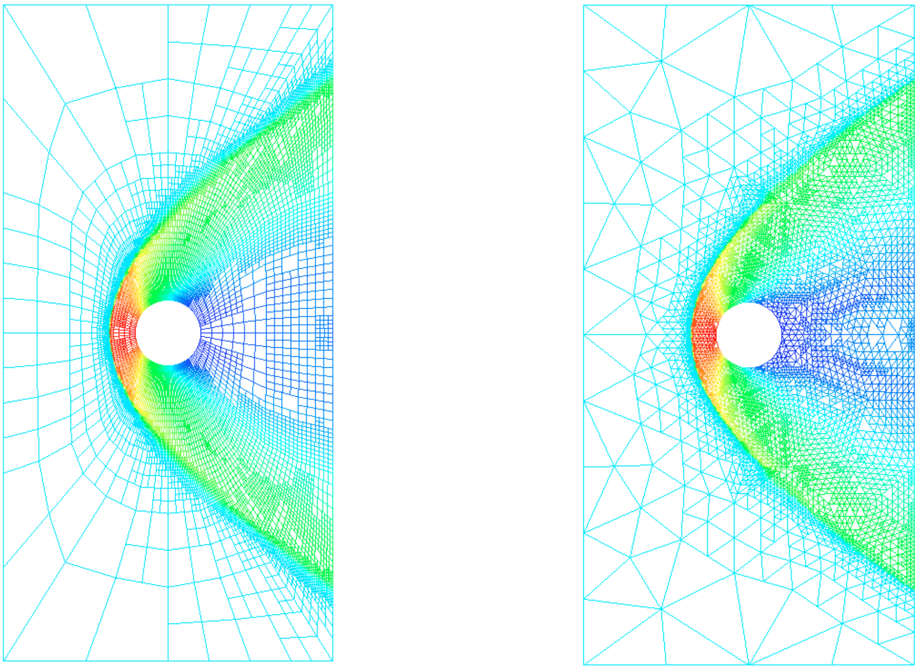
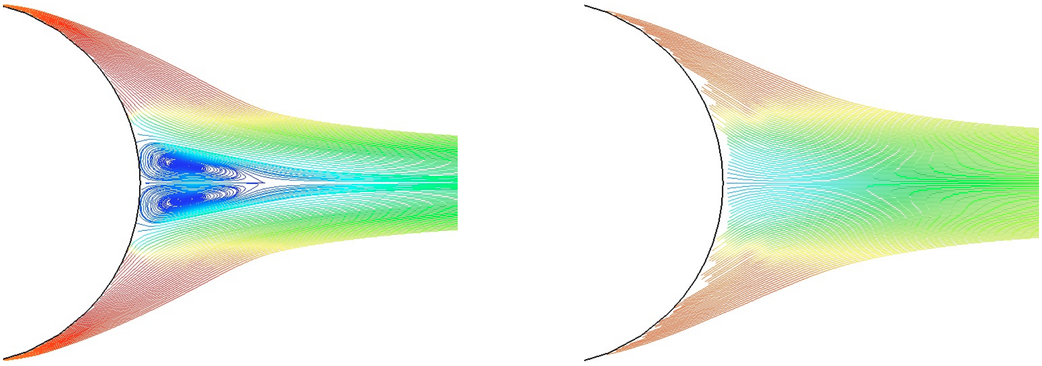


Figure 11. Density streamlines on grid obtained after the seveneth iteration of adaptive refinement procedure with density jump as indicator



(a) (b)

Figure 12. Locally adapted mesh on quadrilaterals (a) and triangles (b)

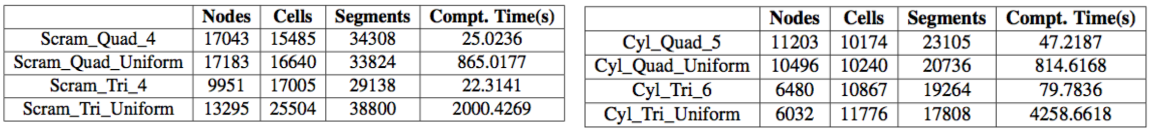


(a) (b)

Figure 13. Streamlines coloured by the density on meshes generated with hierarchical indicator (a) and with jump indicator (b)

Scramjet test case at mach=3

Flow past cylinder test case at mach=3



	Nodes	Cells	Segments	Compt. Time(s)
Scram_Quad_4	17043	15485	34308	25.0236
Scram_Quad_Uniform	17183	16640	33824	865.0177
Scram_Tri_4	9951	17005	29138	22.3141
Scram_Tri_Uniform	13295	25504	38800	2000.4269

	Nodes	Cells	Segments	Compt. Time(s)
Cyl_Quad_5	11203	10174	23105	47.2187
Cyl_Quad_Uniform	10496	10240	20736	814.6168
Cyl_Tri_6	6480	10867	19264	79.7836
Cyl_Tri_Uniform	6032	11776	17808	4258.6618

Table 1

Table 2

Figure 14. Comparison of computational times

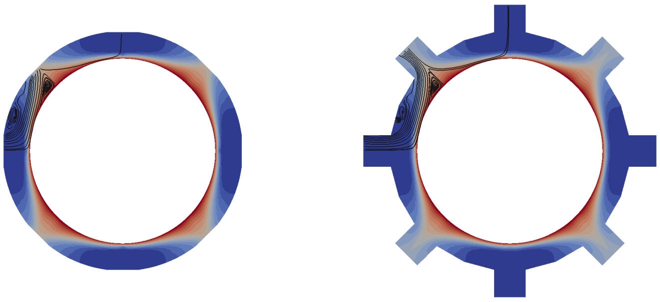
slope limiters for high-order Discontinous Galerkin, low mach number computation with some remarkable approaches.

7. Bilateral Contracts and Grants with Industry

7.1. Optimal (Aerospace Valley)

Participants: Roland Becker, Kossivi Gokpi, Robert Luce, Eric Schall, David Trujillo.

Optimal is a research project related to the cooling of the stator of a turbomachinery. Both physical experiments and numerical simulations are employed. This project has three industrial (Liebherr, Epsilon, and SIBI) and three academic partners (Universities of Pau, Poitiers, and Toulouse). It has been evaluated by the cluster Aerospace Valley. The PhD-thesis of Kossivi Gokpi is financed by this project.



v_{in}	geo1	geo2
10	135,97	130,53
20	216,73	213,59
50	352,61	341,47

Figure 15. Temperature field and recirculation for two geometries and computed Nusselt numbers for different inflow velocities.

Our contributions concern the numerical simulation of the viscous flow in different geometrical configurations. Comparison with experimental data will be investigated with respect to the Nusselt number. The computed temperature and streamlines for typical geometries are shown in Figure 15. In addition, the computed Nusselt numbers for the two configurations and varying inflow velocities are given.

Among the different questions concerning modeling such as the boundary conditions at the in- and outlets and the sensitivity to the geometry, a particular point of interest is the study of compressibility effects.

The experimental part of the product is conducted in collaboration with Mathieu Mory, professor at UPPA, and the post-doctoral position of Stéphane Soubacq, who started to work in 10/2009, is financed by the project. The modeling and numerical simulation is done in collaboration with Abdellah Saboni, professor at UPPA.

7.2. Fractured reservoir (Total)

Participants: Robert Luce, David Trujillo.

We have developped specific meshing tools in order to take into account the interaction between faults and a petroleum reservoir for the company Total. This work was done in collaboration with Eric Dubach and Pierre Puiseux from LMA.

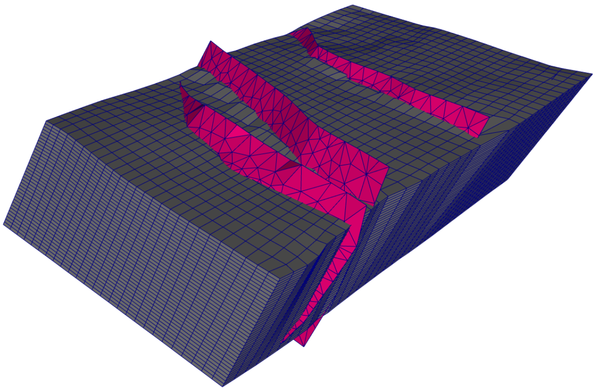


Figure 16. Fractured reservoir

8. Dissemination

8.1. Teaching - Supervision - Juries

8.1.1. Teaching

The LMA has proposed a new Master program starting in 2007, which is called MMS (Mathématiques, Modélisation et Simulation) and has a focus on analysis, modeling, and numerical computations in PDEs; Robert Luce and R. Becker are co-responsables of this Master program. The core of this education is formed by lectures in four fields : PDE-theory, mechanics, numerical analysis, and simulation tools.

This master program includes lectures on physical applications, one of the three proposed application fields is CFD; lectures are provided by the members of the project; especially the following lectures have been given:

Simulation numérique 1, Robert Luce and Eric Dubach,
Analyse numérique des EDP, R. Becker and D. Capatina,
Simulation numérique 2, Robert Luce and Eric Dubach,
Méthodes numériques pour les EDP, R. Becker,
Mécanique des fluides, R. Becker,
Simulation numérique 3, P. Puiseux
Mécanique des Fluides et Turbulence, Eric Schall, D. Graebling

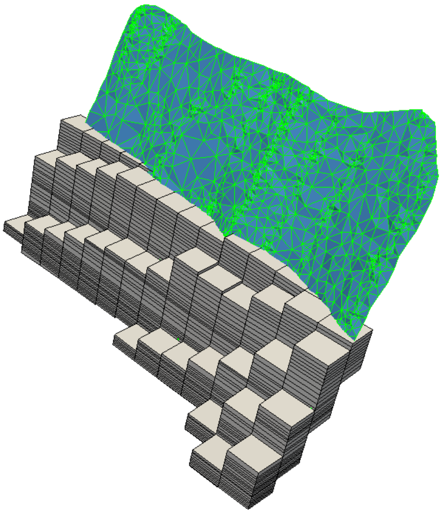


Figure 17. Intersection between fault and reservoir

9. Bibliography

Major publications by the team in recent years

[1] C. AMROUCHE, N. E. H. SELOULA. *Lp-Theory for Vector Potentials and Sobolev's Inequalities for Vector Fields*, 2011, 7 pages, http://hal.inria.fr/hal-00629127/en.

[2] C. AMROUCHE, N. E. H. SELOULA. *Stokes Equations and Elliptic Systems With Non Standard Boundary Conditions*, 2011, 8 pages, http://hal.inria.fr/hal-00629131/en.

[3] R. BECKER. *Adaptive Finite Element Methods for incompressible flow problems*, in "16th Int. Conf. on Finite Elements for Flow Problems", München, Germany, November 2011, http://hal.inria.fr/hal-00646745/en.

[4] R. BECKER. *Adaptive Finite Elements for sensitivity computations*, in "Workshop on Discretization methods for fluid flows", Marseille, France, September 2011, http://hal.inria.fr/hal-00646746/en.

[5] R. BECKER, M. BRAACK. *A Finite Element Pressure Gradient Stabilization for the Stokes Equations Based on Local Projections*, in "Calcolo", 2001, vol. 38, n$^{\mathrm{o}}$ 4, p. 173–199.

[6] R. BECKER, E. BURMAN, P. HANSBO. *A finite element time relaxation method*, in "Comptes Rendus de l Académie des Sciences - Series I - Mathematics", 2011, vol. 349, n$^{\mathrm{o}}$ 5-6, p. 353-356, http://hal.inria.fr/hal-00645159/en.

[7] R. BECKER, E. BURMAN, P. HANSBO. *A hierarchical NXFEM for fictitious domain simulations*, in "International Journal for Numerical Methods in Engineering", 2011, vol. 4-5, p. 549-559, http://hal.inria.fr/hal-00645157/en.

[8] R. Becker, D. Capatina-Papaghiuc. *Numerical analysis of a matrix-valued transport equation with applications in non-Newtonian flows*, in "7th ICIAM", Vancouver, Canada, November 2011, http://hal.inria.fr/hal-00646530/en.

[9] R. Becker, D. Capatina-Papaghiuc, D. Graebling, J. Joie. *Nonconforming finite element approximation of the Giesekus model for polymer flows*, in "Computers and Fluids", 2011, vol. 46, p. 142 - 147, http://hal.inria.fr/hal-00645152/en.

[10] R. Becker, D. Capatina-Papaghiuc, D. Graebling, J. Joie. *Robust approximation of Giesekus flow by nonconforming finite elements*, in "16th Int. Conf. on Finite Elements for Flow Problems", Münich, Germany, November 2011, http://hal.inria.fr/hal-00646535/en.

[11] R. Becker, D. Capatina-Papaghiuc, J. Joie. *Connections between discontinuous Galerkin and nonconforming finite element methods for the Stokes equations*, in "Numerical Methods for Partial Differential Equations / Numerical Methods for Partial Differential Equations An International Journal", March 2011, n$^{\mathrm{o}}$ DOI: 10.1002/num.20671 [*DOI :* 10.1002/NUM.20671], http://hal.inria.fr/inria-00537872/en.

[12] R. Becker, D. Capatina-Papaghiuc, R. Luce. *A posteriori error estimators based on H(div)- reconstruction for diffusion-convection-reaction equation*, in "9th Enumath", Leicester, United Kingdom, November 2011, http://hal.inria.fr/hal-00646537/en.

[13] R. Becker, D. Capatina-Papaghiuc, R. Luce, D. Trujillo. *A posteriori error estimation for sensitivity analysis in finite element methods*, in "11th US National Congress on Computational Mechanics", Minneapolis, United States, November 2011, http://hal.inria.fr/hal-00646533/en.

[14] R. Becker, K. Gokpi, É. Schall, D. Trujillo. *A posteriori error estimators for grid adaptation with Galerkin discontinuous finite element method*, in "8th Int. Conf. on Heat Transfer, Fluid Mechanics and Thermodynamics (HEFAT)", Pointe Aux Piments, Mauritius, November 2011, http://hal.inria.fr/hal-00646856/en.

[15] R. Becker, K. Gokpi, É. Schall, D. Trujillo. *Comparison of two types of a posteriori error estimators on mesh adaptation in discontinuous Galerkin finite elements methods*, in "4th European Conference for Aerospace Sciences (EUCASS)", St. Petersburg, Russian Federation, November 2011, http://hal.inria.fr/hal-00646853/en.

[16] R. Becker, S. Mao. *Quasi-optimality of adaptive non-conforming finite element methods for the Stokes equations*, in "SIAM Journal on Numerical Analysis", 2011, vol. 49, n$^{\mathrm{o}}$ 3, p. 970-991, http://hal.inria.fr/hal-00645150/en.

[17] R. Becker, R. Rannacher. *An Optimal Control Approach to A-Posteriori Error Estimation*, in "Acta Numerica 2001", A. Iserles (editor), Cambridege University Press, 2001, p. 1–102.

[18] R. Becker, R. Rannacher. *A feed-back approach to error control in finite element methods: Basic analysis and examples*, in "East-West J. Numer. Math.", 1996, vol. 4, p. 237–264.

[19] R. Becker, D. Trujillo. *A remark on the optimality of adaptive finite element methods*, in "Comptes Rendus de l Académie des Sciences - Series I - Mathematics", 2011, vol. 349, p. 225-228, http://hal.inria.fr/hal-00645158/en.

[20] R. BECKER, D. TRUJILLO. *Concepts of the finite element library Concha*, in "Monografias Matematicas Garcia de Galdeano", December 2011, vol. 35, p. 59-67, http://hal.inria.fr/hal-00649001/en.

[21] R. BECKER, D. TRUJILLO, E. ESTECAHANDY. *Weighted marking for goal-oriented adaptive finite element methods*, in "SIAM Journal on Numerical Analysis", 2011, http://hal.inria.fr/hal-00647356/en.

[22] R. BECKER, B. VEXLER. *Mesh Refinement and Numerical Sensitivity Analysis for Parameter Calibration of Partial Differential Equations*, in "J. Comput. Phys.", 2005, vol. 206, n^{o} 1, p. 95-110.

[23] R. BECKER, S. MAO, Z.-C. SHI. *A convergent adaptive finite element method with optimal complexity*, in "Electronic Transactions on Numerical Analysis", 2008, http://hal.inria.fr/inria-00343020/en/.

[24] D. BRAESS, R. HOPPE, J. SCHOBERL. *A A posteriori estimators for obstacle problems by the hypercircle method*, in "Comput. Vis. Sci.", 2008, vol. 11, n^{o} 4-6, p. 351-362.

[25] D. CAPATINA-PAPAGHIUC, N. BARRAU. *Numerical simulation of anisothermal flows of Newtonian fluids*, in "Monografias Matematicas Garcia de Galdeano", 2011, vol. 35, p. 37-46, http://hal.inria.fr/hal-00646561/en.

[26] D. CAPATINA-PAPAGHIUC. *A positivity preserving discontinuous Galerkin method with applications in polymer flows*, in "Int. Conf. " Mathematical Fluid Mechanics and Biomedical Applications"", Ponta Delgada, Portugal, November 2011, http://hal.inria.fr/hal-00646529/en.

[27] D. CAPATINA-PAPAGHIUC. *Analyse de méthodes mixtes d'éléments finis en mécanique*, Université de Pau et des Pays de l'Adour, November 2011, Habilitation à Diriger des Recherches, http://hal.inria.fr/tel-00647026/en.

[28] D. CAPATINA-PAPAGHIUC. *Numerical analysis of a Riccati type matrix transport equation*, in "7th Workshop "Variational Multiscale Methods"", Glasgow, United Kingdom, November 2011, http://hal.inria.fr/hal-00646527/en.

[29] D. CAPATINA-PAPAGHIUC, J.-M. THOMAS. *Nonconforming finite element methods without numerical locking.*, in "Numer. Math.", 1998, vol. 81, n^{o} 2, p. 163-186.

[30] M. LI, R. BECKER, S. MAO. *A remark on supercloseness and extrapolation of the quadrilateral han element for the stokes equations*, in "Comptes Rendus de l Académie des Sciences - Series I - Mathematics", 2011, vol. 349, n^{o} 17-18, p. 1017 - 1020, http://hal.inria.fr/hal-00645148/en.

[31] R. LUCE, B. WOHLMUTH. *A local a posteriori error estimator based on equilibrated fluxes.*, in "SIAM J. Numer. Anal.", 2004, vol. 42, n^{o} 4, p. 1394-1414.

[32] E. SCHALL, C. VIOZAT, B. KOOBUS, A. DERVIEUX. *Computation of low Mach thermical flows with implicit upwind methods.*, in "Int. J. Heat Mass Transfer", 2003, vol. 46, n^{o} 20, p. 3909-3926.

[33] J.-M. THOMAS, D. TRUJILLO. *Mixed finite volume methods.*, in "Int. J. Numer. Methods Engrg.", 1999, vol. 46, n^{o} 9, p. 1351-1366.

[34] C. XIONG, Y. LI. *A posteriori error estimators for optimal distributed control governed by the first-order linear hyperbolic equation: DG method*, in "Numerical Methods for Partial Differential Equations", 2011, vol. 27, n^{o} 3, p. 491-506, http://hal.inria.fr/hal-00646952/en.

[35] C. XIONG, Y. LI. *Error analysis for optimal control problem governed by convection diffusion equations: DG method*, in "Journal of Computational and Applied Mathematics", 2011, vol. 235, n^{o} 10, p. 3163-3177, http://hal.inria.fr/hal-00646954/en.

Publications of the year

Articles in International Peer-Reviewed Journals

[36] R. BECKER, D. CAPATINA-PAPAGHIUC, J. JOIE. *Connections between discontinuous Galerkin and non-conforming finite element methods for the Stokes equations*, in "Numerical Methods for Partial Differential Equations / Numerical Methods for Partial Differential Equations An International Journal", 2012, vol. 28, n^{o} 3, p. 1013-1041 [*DOI :* 10.1002/NUM.20671], http://hal.inria.fr/inria-00537872.

[37] R. BECKER, K. GOKPI, E. SCHALL, D. TRUJILLO. *Comparison of hierarchical and non-hierarchical error indicators for adaptive mesh refinement for the Euler equations*, in "Journal of Aerospace Engineering", 2012, http://hal.inria.fr/hal-00766918.

[38] R. BECKER, K. GOKPI, É. SCHALL, D. TRUJILLO. *Fully implicit adaptive method using discontinuous Galerkin finite elements for high speed flows*, in "Int. J. Aerodynamics", 2012, vol. 2, http://hal.inria.fr/hal-00766915.

[39] R. BECKER, D. TRUJILLO. *Concepts of the Finite Element Library Concha*, in "Monografias Matematica", 2012, p. 59-67, http://hal.inria.fr/hal-00766922.

[40] D. CAPATINA-PAPAGHIUC, N. BARRAU. *Numerical simulation of anisothermal flows of Newtonian fluids*, in "Monografias Matematicas Garcia de Galdeano", 2012, vol. 35, p. 37-46, http://hal.inria.fr/hal-00646561.

International Conferences with Proceedings

[41] N. BARRAU, D. CAPATINA-PAPAGHIUC. *Numerical simulation of anisothermal flows of Newtonian fluids*, in "11th International Conference Zaragoza-Pau on Applied Mathematics and Statistics", Jaca, Spain, Prensas Univ. Zaragoza, Zaragoza, 2012, vol. 37, p. 37-46, http://hal.inria.fr/inria-00539640.

[42] R. BECKER, D. CAPATINA-PAPAGHIUC, R. LUCE. *A unified approach to build robust H(div)-reconstructed estimators on primal mesh*, in "ECCOMAS.", Vienne, Austria, 2012, http://hal.inria.fr/hal-00766876.

References in notes

[43] D. ARNOLD. *An interior penalty finite element method with discontinuous elements*, in "SIAM J. Numer. Anal.", 1982, vol. 19, p. 742-760.

[44] G. BAKER. *Finite element methods for elliptic equations using nonconforming elements*, in "Math. Comp.", 1977, vol. 31, p. 45-59.

[45] F. BASSI, S. REBAY. *High-order accurate discontinuous finite element solution of the 2D Euler equations*, in "J. Comput. Phys.", 1997, vol. 138, n^{o} 2, p. 251–285.

[46] R. BECKER, E. BURMAN, P. HANSBO. *A Nitsche extended finite element method for incompressible elasticity with discontinuous modulus of elasticity*, in "Comput. Methods Appl. Mech. Engrg.", 2009, vol. 198, n^{o} 41-44, p. 3352-3360, http://hal.inria.fr/inria-00437190/en/.

[47] R. BECKER, E. BURMAN, P. HANSBO. *A hierarchical nxfem for fictitious domain simulations*, in "Int. J. Numer. Meth. Engng", 2010, to appear, http://hal.inria.fr/inria-00539171/en.

[48] R. BECKER, P. HANSBO. *A simple pressure stabilization method for the Stokes equation*, in "Comm. Numer. Methods Eng.", 2008, vol. 24, n^{o} 11, p. 1421-1430.

[49] R. BECKER, C. JOHNSON, R. RANNACHER. *Adaptive Error Control for Multigrid Finite Element Methods*, in "Computing", 1995, vol. 55, p. 271-288.

[50] R. BECKER, S. MAO. *An optimally convergent adaptive mixed finite element method*, in "Numer. Math.", 2008, vol. 111, n^{o} 1, p. 35-54.

[51] R. BECKER, S. MAO. *Convergence and quasi-optimal complexity of a simple adaptive finite element method*, in "M2AN", 2009, vol. 43, p. 1203–1219, http://dx.doi.org/10.1051/m2an/2009036.

[52] R. BECKER, R. RANNACHER. *Weighted a posteriori error control in FE methods*, in "ENUMATH'97", H. G. BOCK, ET AL. (editors), World Sci. Publ., Singapore, 1995.

[53] R. BECKER, S. MAO, Z. SHI. *A convergent nonconforming adaptive finite element method with optimal complexity*, in "SIAM Journal on Numerical Analysis", 2010, vol. 47, p. 4639–4659, http://hal.inria.fr/inria-00438541/en/.

[54] P. BINEV, W. DAHMEN, R. DEVORE. *Adaptive finite element methods with convergence rates.*, in "Numer. Math.", 2004, vol. 97, n^{o} 2, p. 219-268.

[55] M. BRAACK, E. BURMAN. *Local Projection Stabilization for the Oseen Problem and its Interpretation as a Variational Multiscale Method*, in "SIAM J. Numer. Anal.", 2006, vol. 43, n^{o} 6, p. 2544-2566.

[56] D. BRAESS, R. HOPPE, J. SCHOBERL. *A A posteriori estimators for obstacle problems by the hypercircle method*, in "Comput. Vis. Sci.", 2008, vol. 11, n^{o} 4-6, p. 351-362.

[57] A. BROOKS, T. HUGHES. *Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations.*, in "Comput. Methods Appl. Mech. Comput. Methods Appl. Mech. Engrg.", 1982, vol. 32, p. 199-259.

[58] E. BURMAN, M. A. FERNÁNDEZ. *Galerkin finite element methods with symmetric pressure stabilization for the transient Stokes equations: stability and convergence analysis*, in "SIAM J. Numer. Anal.", 2008, vol. 47, n^{o} 1, p. 409–439, http://dx.doi.org/10.1137/070707403.

[59] E. BURMAN, P. HANSBO. *Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems.*, in "Comput. Methods Appl. Mech. Engrg.", 2004, vol. 193, n^{o} 15-16, p. 1437-1453.

[60] E. DUBACH, R. LUCE, J. THOMAS. *Pseudo-conform polynomial Lagrange finite elements on quadrilaterals and hexahedra.*, in "Comm. Pure Appl. Anal.", 2009, vol. 8, p. 237-254, http://hal.inria.fr/inria-00438537/en/.

[61] E. DUBACH, R. LUCE, J. THOMAS. *Pseudo-conforming polynomial finite element on quadrilaterals*, in "Int. J. Comput. Math.", 2009, vol. 80, n^{o} 10-11, p. 1798-1816, http://hal.inria.fr/inria-00438536/en/.

[62] J.-L. GUERMOND. *Stabilization of Galerkin approximations of transport equations by subgrid modeling*, in "Modél. Math. Anal. Numér.", 1999, vol. 33, n^{o} 6, p. 1293-1316.

[63] A. HANSBO, P. HANSBO. *An unfitted finite element method, based on Nitsche's method, for elliptic interface problems*, in "Comp. Methods Appl. Mech. Engrg. in Applied Mechanics and Engineering", 2002, vol. 191, n^{o} 47-48, p. 537-5552.

[64] A. HANSBO, P. HANSBO. *A finite element method for the simulation of strong and weak discontinuities in solid mechanics.*, in "Comput. Methods Appl. Mech. Eng.", 2004, vol. 193, n^{o} 33-35, p. 3523-3540.

[65] M. A. HULSEN, R. FATTAL, R. KUPFERMAN. *Flow of viscoelastic fluids past a cylinder at high Weissenberg number: Stabilized simulations using matrix logarithms*, in "Journal of Non-Newtonian Fluid Mechanics", 2005, vol. 127, n^{o} 1, p. 27–39, http://www.sciencedirect.com/science/article/B6TGV-4G53W93-1/2/fe0f91e467f09e43fa76f6896404184b.

[66] C. JOHNSON, J. PITKÄRANTA. *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, in "Math. Comp.", 1986, vol. 46, p. 1-26.

[67] C. JOHNSON, A. SZEPESSY, P. HANSBO. *On the convergence of shock-capturing streamline diffusion finite element methods for hyperbolic conservation laws.*, in "Math. Comp.", 1990, vol. 54, n^{o} 189, p. 107-129.

[68] P. LESAINT, P. RAVIART. *On a finite element method for solving the Neutron transport equation*, in "Mathematical Aspects of Finite Elements in Partial Differential Equations", C. DE BOOR (editor), Academic Press, New York, 1974.

[69] P. MORIN, R. H. NOCHETTO, K. G. SIEBERT. *Data oscillation and convergence of adaptive FEM.*, in "SIAM J. Numer. Anal.", 2000, vol. 38, n^{o} 2, p. 466-488.

[70] M. SCHÄFER, S. TUREK. *Benchmark computations of laminar flow around a cylinder. (With support by F. Durst, E. Krause and R. Rannacher)*, in "Flow Simulation with High-Performance Computers II. DFG priority research program results 1993-1995", E. HIRSCHEL (editor), Notes Numer. Fluid Mech., Vieweg, Wiesbaden, 1996, n^{o} 52, p. 547-566.

[71] R. STEVENSON. *Optimality of a standard adaptive finite element method*, in "Found. Comput. Math.", 2007, vol. 7, n^{o} 2, p. 245-269.