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Project-Team REGULARITY

Probabilistic modelling of irregularity and
application to uncertainties management

IN COLLABORATION WITH: Laboratoire de Mathématiques Appliquées aux Systèmes (MAS)

RESEARCH CENTER
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THEME
Stochastic Methods and Models

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Project-Team REGULARITY

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2. Overall Objectives

2.1. Overall Objectives

Many phenomena of interest are analyzed and controlled through graphs or n-dimensional images. Often, these graphs have an *irregular aspect*, whether the studied phenomenon is of natural or artificial origin. In the first class, one may cite natural landscapes, most biological signals and images (EEG, ECG, MR images, ...), and temperature records. In the second class, prominent examples include financial logs and TCP traces.

Such irregular phenomena are usually not adequately described by purely deterministic models, and a probabilistic ingredient is often added. Stochastic processes allow to take into account, with a firm theoretical basis, the numerous microscopic fluctuations that shape the phenomenon.

In general, it is a wrong view to believe that irregularity appears as an epiphenomenon, that is conveniently dealt with by introducing randomness. In many situations, and in particular in some of the examples mentioned above, irregularity is a core ingredient that cannot be removed without destroying the phenomenon itself. In some cases, irregularity is even a necessary condition for proper functioning. A striking example is that of ECG: an ECG is inherently irregular, and, moreover, in a mathematically precise sense, an *increase* in its regularity is strongly correlated with a *degradation* of its condition.

In fact, in various situations, irregularity is a crucial feature that can be used to assess the behaviour of a given system. For instance, irregularity may be the result of two or more sub-systems that act in a concurrent way to achieve some kind of equilibrium. Examples of this abound in nature (*e.g.* the sympathetic and parasympathetic systems in the regulation of the heart). For artifacts, such as financial logs and TCP traffic, irregularity is in a sense an unwanted feature, since it typically makes regulations more complex. It is again, however, a necessary one. For instance, efficiency in financial markets requires a constant flow of information among agents, which manifests itself through permanent fluctuations of the prices: irregularity just reflects the evolution of this information.

The aim of *Regularity* is to develop a coherent set of methods allowing to model such “essentially irregular” phenomena in view of managing the uncertainties entailed by their irregularity.

Indeed, essential irregularity makes it more difficult to study phenomena in terms of their description, modeling, prediction and control. It introduces *uncertainties* both in the measurements and the dynamics. It is, for instance, obviously easier to predict the short time behaviour of a smooth (*e.g.* C^1) process than of a nowhere differentiable one. Likewise, sampling rough functions yields less precise information than regular ones. As a consequence, when dealing with essentially irregular phenomena, uncertainties are fundamental in the sense that one cannot hope to remove them by a more careful analysis or a more adequate modeling. The study of such phenomena then requires to develop specific approaches allowing to manage in an efficient way these inherent uncertainties.

2.2. Highlights of the Year

- Release of version 2.1 of the software toolbox FracLab.

BEST PAPER AWARD :

[39] **A New Method for Multifractal Spectrum Estimation with Applications to Texture Description in International Conference on Mass Data Analysis of Images and Signals (MDA’2012).** J. LÉVY-VÉHEL, M. TESMER.

3. Scientific Foundations

3.1. Theoretical aspects: probabilistic modeling of irregularity

The modeling of essentially irregular phenomena is an important challenge, with an emphasis on understanding the sources and functions of this irregularity. Probabilistic tools are well-adapted to this task, provided one can design stochastic models for which the regularity can be measured and controlled precisely. Two points deserve special attention:

- first, the study of regularity has to be *local*. Indeed, in most applications, one will want to act on a system based on local temporal or spatial information. For instance, detection of arrhythmias in ECG or of krachs in financial markets should be performed in “real time”, or, even better, ahead of time. In this sense, regularity is a *local* indicator of the *local* health of a system.
- Second, although we have used the term “irregularity” in a generic and somewhat vague sense, it seems obvious that, in real-world phenomena, regularity comes in many colors, and a rigorous analysis should distinguish between them. As an example, at least two kinds of irregularities are present in financial logs: the local “roughness” of the records, and the local density and height of jumps. These correspond to two different concepts of regularity (in technical terms, Hölder exponents and local index of stability), and they both contribute a different manner to financial risk.

In view of the above, the *Regularity* team focuses on the design of methods that:

1. define and study precisely various relevant measures of local regularity,
2. allow to build stochastic models versatile enough to mimic the rapid variations of the different kinds of regularities observed in real phenomena,
3. allow to estimate as precisely and rapidly as possible these regularities, so as to alert systems in charge of control.

Our aim is to address the three items above through the design of mathematical tools in the field of probability (and, to a lesser extent, statistics), and to apply these tools to uncertainty management as described in the following section. We note here that we do not intend to address the problem of controlling the phenomena based on regularity, that would naturally constitute an item 4 in the list above. Indeed, while we strongly believe that generic tools may be designed to measure and model regularity, and that these tools may be used to analyze real-world applications, in particular in the field of uncertainty management, it is clear that, when it comes to control, application-specific tools are required, that we do not wish to address.

The research topics of the *Regularity* team can be roughly divided into two strongly interacting axes, corresponding to two complementary ways of studying regularity:

1. developments of tools allowing to characterize, measure and estimate various notions of local regularity, with a particular emphasis on the stochastic frame,
2. definition and fine analysis of stochastic models for which some aspects of local regularity may be prescribed.

These two aspects are detailed in sections 3.2 and 3.3 below.

3.2. Tools for characterizing and measuring regularity

Fractional Dimensions

Although the main focus of our team is on characterizing *local* regularity, on occasions, it is interesting to use a *global* index of regularity. Fractional dimensions provide such an index. In particular, the *regularization dimension*, that was defined in [30], is well adapted to the study stochastic processes, as its definition allows to build robust estimators in an easy way. Since its introduction, regularization dimension has been used by various teams worldwide in many different applications including the characterization of certain stochastic processes, statistical estimation, the study of mammographies or galactograms for breast carcinomas detection, ECG analysis for the study of ventricular arrhythmia, encephalitis diagnosis from EEG, human skin analysis, discrimination between the nature of radioactive contaminations, analysis of porous media textures, well-logs data analysis, agro-alimentary image analysis, road profile analysis, remote sensing, mechanical systems assessment, analysis of video games, ... (see <http://regularity.saclay.inria.fr/theory/localregularity/biblioregdim> for a list of works using the regularization dimension).

Hölder exponents

The simplest and most popular measures of local regularity are the pointwise and local Hölder exponents. For a stochastic process $\{X(t)\}_{t \in \mathbb{R}}$ whose trajectories are continuous and nowhere differentiable, these are defined, at a point t_0 , as the random variables:

$$\alpha_X(t_0, \omega) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{t, u \in B(t_0, \rho)} \frac{|X_t - X_u|}{\rho^\alpha} < \infty \right\}, \quad (1)$$

and

$$\tilde{\alpha}_X(t_0, \omega) = \sup \left\{ \alpha : \limsup_{\rho \rightarrow 0} \sup_{t, u \in B(t_0, \rho)} \frac{|X_t - X_u|}{\|t - u\|^\alpha} < \infty \right\}. \quad (2)$$

Although these quantities are in general random, we will omit as is customary the dependency in ω and X and write $\alpha(t_0)$ and $\tilde{\alpha}(t_0)$ instead of $\alpha_X(t_0, \omega)$ and $\tilde{\alpha}_X(t_0, \omega)$.

The random functions $t \mapsto \alpha_X(t_0, \omega)$ and $t \mapsto \tilde{\alpha}_X(t_0, \omega)$ are called respectively the pointwise and local Hölder functions of the process X .

The pointwise Hölder exponent is a very versatile tool, in the sense that the set of pointwise Hölder functions of continuous functions is quite large (it coincides with the set of lower limits of sequences of continuous functions [5]). In this sense, the pointwise exponent is often a more precise tool (*i.e.* it varies in a more rapid way) than the local one, since local Hölder functions are always lower semi-continuous. This is why, in particular, it is the exponent that is used as a basis ingredient in multifractal analysis (see section 3.2). For certain classes of stochastic processes, and most notably Gaussian processes, it has the remarkable property that, at each point, it assumes an almost sure value [16]. SRP, mBm, and processes of this kind (see sections 3.3 and 3.3) rely on the sole use of the pointwise Hölder exponent for prescribing the regularity.

However, α_X obviously does not give a complete description of local regularity, even for continuous processes. It is for instance insensitive to “oscillations”, contrarily to the local exponent. A simple example in the deterministic frame is provided by the function $x^\gamma \sin(x^{-\beta})$, where γ, β are positive real numbers. This so-called “chirp function” exhibits two kinds of irregularities: the first one, due to the term x^γ is measured by the pointwise Hölder exponent. Indeed, $\alpha(0) = \gamma$. The second one is due to the wild oscillations around 0, to which α is blind. In contrast, the local Hölder exponent at 0 is equal to $\frac{\gamma}{1+\beta}$, and is thus influenced by the oscillatory behaviour.

Another, related, drawback of the pointwise exponent is that it is not stable under integro-differentiation, which sometimes makes its use complicated in applications. Again, the local exponent provides here a useful complement to α , since $\tilde{\alpha}$ is stable under integro-differentiation.

Both exponents have proved useful in various applications, ranging from image denoising and segmentation to TCP traffic characterization. Applications require precise estimation of these exponents.

Stochastic 2-microlocal analysis

Neither the pointwise nor the local exponents give a complete characterization of the local regularity, and, although their joint use somewhat improves the situation, it is far from yielding the complete picture.

A fuller description of local regularity is provided by the so-called *2-microlocal analysis*, introduced by J.M. Bony [50]. In this frame, regularity at each point is now specified by two indices, which makes the analysis and estimation tasks more difficult. More precisely, a function f is said to belong to the *2-microlocal space* $C_{x_0}^{s,s'}$, where $s + s' > 0, s' < 0$, if and only if its $m = [s + s']$ -th order derivative exists around x_0 , and if there exists $\delta > 0$, a polynomial P with degree lower than $[s] - m$, and a constant C , such that

$$\left| \frac{\partial^m f(x) - P(x)}{|x-x_0|^{[s]-m}} - \frac{\partial^m f(y) - P(y)}{|y-x_0|^{[s]-m}} \right| \leq C|x-y|^{s+s'-m}(|x-y| + |x-x_0|)^{-s'-[s]+m}$$

for all x, y such that $0 < |x-x_0| < \delta, 0 < |y-x_0| < \delta$. This characterization was obtained in [23], [31]. See [64], [66] for other characterizations and results. These spaces are stable through integro-differentiation, i.e. $f \in C_x^{s,s'}$ if and only if $f' \in C_x^{s-1,s'}$. Knowing to which space f belongs thus allows to predict the evolution of its regularity after derivation, a useful feature if one uses models based on some kind differential equations. A lot of work remains to be done in this area, in order to obtain more general characterizations, to develop robust estimation methods, and to extend the “2-microlocal formalism”: this is a tool allowing to detect which space a function belongs to, from the computation of the Legendre transform of an auxiliary function known as its *2-microlocal spectrum*. This spectrum provide a wealth of information on the local regularity.

In [16], we have laid some foundations for a stochastic version of 2-microlocal analysis. We believe this will provide a fine analysis of the local regularity of random processes in a direction different from the one detailed for instance in [72]. We have defined random versions of the 2-microlocal spaces, and given almost sure conditions for continuous processes to belong to such spaces. More precise results have also been obtained for Gaussian processes. A preliminary investigation of the 2-microlocal behaviour of Wiener integrals has been performed.

Multifractal analysis of stochastic processes

A direct use of the local regularity is often fruitful in applications. This is for instance the case in RR analysis or terrain modeling. However, in some situations, it is interesting to supplement or replace it by a more global approach known as *multifractal analysis* (MA). The idea behind MA is to group together all points with same regularity (as measured by the pointwise Hölder exponent) and to measure the “size” of the sets thus obtained [27], [51], [60]. There are mainly two ways to do so, a geometrical and a statistical one.

In the geometrical approach, one defines the *Hausdorff multifractal spectrum* of a process or function X as the function: $\alpha \mapsto f_h(\alpha) = \dim \{t : \alpha_X(t) = \alpha\}$, where $\dim E$ denotes the Hausdorff dimension of the set E . This gives a fine measure-theoretic information, but is often difficult to compute theoretically, and almost impossible to estimate on numerical data.

The statistical path to MA is based on the so-called *large deviation multifractal spectrum*:

$$f_g(\alpha) = \liminf_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{\log N_n^\varepsilon(\alpha)}{\log n},$$

where:

$$N_n^\varepsilon(\alpha) = \#\{k : \alpha - \varepsilon \leq \alpha_n^k \leq \alpha + \varepsilon\},$$

and α_n^k is the “coarse grained exponent” corresponding to the interval $I_n^k = [\frac{k}{n}, \frac{k+1}{n}]$, i.e.:

$$\alpha_n^k = \frac{\log |Y_n^k|}{-\log n}.$$

Here, Y_n^k is some quantity that measures the variation of X in the interval I_n^k , such as the increment, the oscillation or a wavelet coefficient.

The large deviation spectrum is typically easier to compute and to estimate than the Hausdorff one. In addition, it often gives more relevant information in applications.

Under very mild conditions (e.g. for instance, if the support of f_g is bounded, [40]) the concave envelope of f_g can be computed easily from an auxiliary function, called the *Legendre multifractal spectrum*. To do so, one basically interprets the spectrum f_g as a rate function in a large deviation principle (LDP): define, for $q \in \mathbb{R}$,

$$S_n(q) = \sum_{k=0}^{n-1} |Y_n^k|^q, \quad (3)$$

with the convention $0^q := 0$ for all $q \in \mathbb{R}$. Let:

$$\tau(q) = \liminf_{n \rightarrow \infty} \frac{\log S_n(q)}{-\log(n)}.$$

The Legendre multifractal spectrum of X is defined as the Legendre transform τ^* of τ :

$$f_l(\alpha) := \tau^*(\alpha) := \inf_{q \in \mathbb{R}} (q\alpha - \tau(q)).$$

To see the relation between f_g and f_l , define the sequence of random variables $Z_n := \log |Y_n^k|$ where the randomness is through a choice of k uniformly in $\{0, \dots, n-1\}$. Consider the corresponding moment generating functions:

$$c_n(q) := -\frac{\log E_n[\exp(qZ_n)]}{\log(n)}$$

where E_n denotes expectation with respect to P_n , the uniform distribution on $\{0, \dots, n-1\}$. A version of Gärtner-Ellis theorem ensures that if $\lim c_n(q)$ exists (in which case it equals $1 + \tau(q)$), and is differentiable, then $c^* = f_g - 1$. In this case, one says that the *weak multifractal formalism* holds, i.e. $f_g = f_l$. In favorable cases, this also coincides with f_h , a situation referred to as the *strong multifractal formalism*.

Multifractal spectra subsume a lot of information about the distribution of the regularity, that has proved useful in various situations. A most notable example is the strong correlation reported recently in several works between the narrowing of the multifractal spectrum of ECG and certain pathologies of the heart [61], [63]. Let us also mention the multifractality of TCP traffic, that has been both observed experimentally and proved on simplified models of TCP [2], [47].

Another colour in local regularity: jumps

As noted above, apart from Hölder exponents and their generalizations, at least another type of irregularity may sometimes be observed on certain real phenomena: discontinuities, which occur for instance on financial logs and certain biomedical signals. In this frame, it is of interest to supplement Hölder exponents and their extensions with (at least) an additional index that measures the local intensity and size of jumps. This is a topic we intend to pursue in full generality in the near future. So far, we have developed an approach in the particular frame of *multistable processes*. We refer to section 3.3 for more details.

3.3. Stochastic models

The second axis in the theoretical developments of the *Regularity* team aims at defining and studying stochastic processes for which various aspects of the local regularity may be prescribed.

Multifractional Brownian motion

One of the simplest stochastic process for which some kind of control over the Hölder exponents is possible is probably fractional Brownian motion (fBm). This process was defined by Kolmogorov and further studied by Mandelbrot and Van Ness, followed by many authors. The so-called “moving average” definition of fBm reads as follows:

$$Y_t = \int_{-\infty}^0 \left[(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right] \cdot \mathbb{W}(du) + \int_0^t (t-u)^{H-\frac{1}{2}} \cdot \mathbb{W}(du),$$

where \mathbb{W} denotes the real white noise. The parameter H ranges in $(0, 1)$, and it governs the pointwise regularity: indeed, almost surely, at each point, both the local and pointwise Hölder exponents are equal to H .

Although varying H yields processes with different regularity, the fact that the exponents are constant along any single path is often a major drawback for the modeling of real world phenomena. For instance, fBm has often been used for the synthesis natural terrains. This is not satisfactory since it yields images lacking crucial features of real mountains, where some parts are smoother than others, due, for instance, to erosion.

It is possible to generalize fBm to obtain a Gaussian process for which the pointwise Hölder exponent may be tuned at each point: the *multifractional Brownian motion (mBm)* is such an extension, obtained by substituting the constant parameter $H \in (0, 1)$ with a *regularity function* $H : \mathbb{R}_+ \rightarrow (0, 1)$.

mBm was introduced independently by two groups of authors: on the one hand, Peltier and Levy-Vehel [28] defined the mBm $\{X_t; t \in \mathbb{R}_+\}$ from the moving average definition of the fractional Brownian motion, and set:

$$X_t = \int_{-\infty}^0 \left[(t-u)^{H(t)-\frac{1}{2}} - (-u)^{H(t)-\frac{1}{2}} \right] \cdot \mathbb{W}(du) + \int_0^t (t-u)^{H(t)-\frac{1}{2}} \cdot \mathbb{W}(du),$$

On the other hand, Benassi, Jaffard and Roux [49] defined the mBm from the harmonizable representation of the fBm, *i.e.*:

$$X_t = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+\frac{1}{2}}} \cdot \widehat{\mathbb{W}}(d\xi),$$

where $\widehat{\mathbb{W}}$ denotes the complex white noise.

The Hölder exponents of the mBm are prescribed almost surely: the pointwise Hölder exponent is $\alpha_X(t) = H(t) \wedge \alpha_H(t)$ a.s., and the local Hölder exponent is $\tilde{\alpha}_X(t) = H(t) \wedge \tilde{\alpha}_H(t)$ a.s. Consequently, the regularity of the sample paths of the mBm are determined by the function H or by its regularity. The multifractional Brownian motion is our prime example of a stochastic process with prescribed local regularity.

The fact that the local regularity of mBm may be tuned *via* a functional parameter has made it a useful model in various areas such as finance, biomedicine, geophysics, image analysis, A large number of studies have been devoted worldwide to its mathematical properties, including in particular its local time. In addition, there is now a rather strong body of work dealing the estimation of its functional parameter, *i.e.* its local regularity. See <http://regularity.saclay.inria.fr/theory/stochasticmodels/bibliombm> for a partial list of works, applied or theoretical, that deal with mBm.

Self-regulating processes

We have recently introduced another class of stochastic models, inspired by mBm, but where the local regularity, instead of being tuned “exogenously”, is a function of the amplitude. In other words, at each point t , the Hölder exponent of the process X verifies almost surely $\alpha_X(t) = g(X(t))$, where g is a fixed deterministic function verifying certain conditions. A process satisfying such an equation is generically termed a *self-regulating process* (SRP). The particular process obtained by adapting adequately mBm is called the self-regulating multifractional process [3]. Another instance is given by modifying the Lévy construction of Brownian motion [42]. The motivation for introducing self-regulating processes is based on the following general fact: in nature, the local regularity of a phenomenon is often related to its amplitude. An intuitive example is provided by natural terrains: in young mountains, regions at higher altitudes are typically more irregular than regions at lower altitudes. We have verified this fact experimentally on several digital elevation models [7]. Other natural phenomena displaying a relation between amplitude and exponent include temperatures records and RR intervals extracted from ECG [38].

To build the SRMP, one starts from a field of fractional Brownian motions $B(t, H)$, where (t, H) span $[0, 1] \times [a, b]$ and $0 < a < b < 1$. For each fixed H , $B(t, H)$ is a fractional Brownian motion with exponent H . Denote:

$$\overline{X}_{\alpha'}^{\beta'} = \alpha' + (\beta' - \alpha') \frac{X - \min_K(X)}{\max_K(X) - \min_K(X)}$$

the affine rescaling between α' and β' of an arbitrary continuous random field over a compact set K . One considers the following (stochastic) operator, defined almost surely:

$$\begin{aligned} \Lambda_{\alpha', \beta'} : \mathcal{C}([0, 1], [\alpha, \beta]) &\rightarrow \mathcal{C}([0, 1], [\alpha, \beta]) \\ Z(\cdot) &\mapsto \overline{B(\cdot, g(Z(\cdot)))}_{\alpha'}^{\beta'} \end{aligned}$$

where $\alpha \leq \alpha' < \beta' \leq \beta$, α and β are two real numbers, and α', β' are random variables adequately chosen. One may show that this operator is contractive with respect to the sup-norm. Its unique fixed point is the SRMP. Additional arguments allow to prove that, indeed, the Hölder exponent at each point is almost surely $g(t)$.

An example of a two dimensional SRMP with function $g(x) = 1 - x^2$ is displayed on figure 1.

We believe that SRP open a whole new and very promising area of research.

Multistable processes

Non-continuous phenomena are commonly encountered in real-world applications, *e.g.* financial records or EEG traces. For such processes, the information brought by the Hölder exponent must be supplemented by some measure of the density and size of jumps. Stochastic processes with jumps, and in particular Lévy processes, are currently an active area of research.

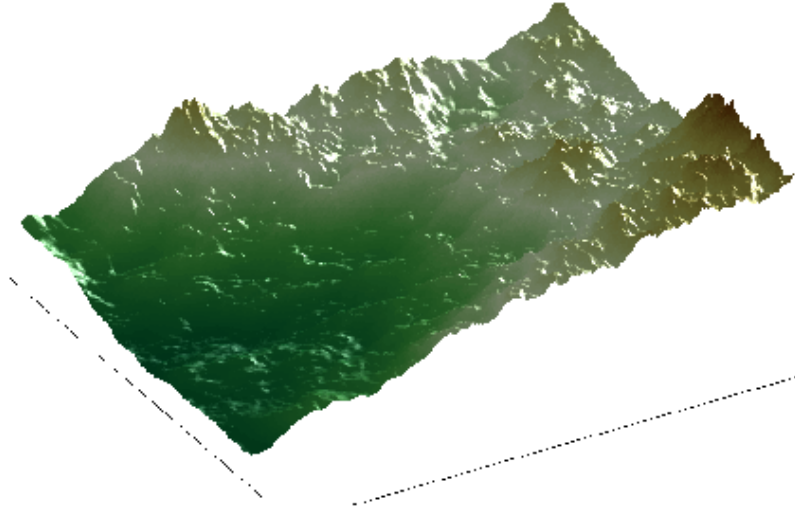


Figure 1. Self-regulating multifractional process with $g(x) = 1 - x^2$

The simplest class of non-continuous Lévy processes is maybe the one of stable processes [74]. These are mainly characterized by a parameter $\alpha \in (0, 2]$, the *stability index* ($\alpha = 2$ corresponds to the Gaussian case, that we do not consider here). This index measures in some precise sense the intensity of jumps. Paths of stable processes with α close to 2 tend to display “small jumps”, while, when α is near 0, their aspect is governed by large ones.

In line with our quest for the characterization and modeling of various notions of local regularity, we have defined *multistable processes*. These are processes which are “locally” stable, but where the stability index α is now a function of time. This allows to model phenomena which, at times, are “almost continuous”, and at others display large discontinuities. Such a behaviour is for instance obvious on almost any sufficiently long financial record.

More formally, a multistable process is a process which is, at each time u , tangent to a stable process [59]. Recall that a process Y is said to be tangent at u to the process Y'_u if:

$$\lim_{r \rightarrow 0} \frac{Y(u + rt) - Y(u)}{r^h} = Y'_u(t), \quad (4)$$

where the limit is understood either in finite dimensional distributions or in the stronger sense of distributions. Note Y'_u may and in general will vary with u .

One approach to defining multistable processes is similar to the one developed for constructing mBm [28]: we consider fields of stochastic processes $X(t, u)$, where t is time and u is an independent parameter that controls the variation of α . We then consider a “diagonal” process $Y(t) = X(t, t)$, which will be, under certain conditions, “tangent” at each point t to a process $t \mapsto X(t, u)$.

A particular class of multistable processes, termed “linear multistable multifractional motions” (lmmm) takes the following form [9], [8]. Let (E, \mathcal{E}, m) be a σ -finite measure space, and Π be a Poisson process on $E \times \mathbb{R}$ with mean measure $m \times \mathcal{L}$ (\mathcal{L} denotes the Lebesgue measure). An lmmm is defined as:

$$Y(t) = a(t) \sum_{(X,Y) \in \Pi} \Upsilon^{<-1/\alpha(t)>} \left(|t - X|^{h(t)-1/\alpha(t)} - |X|^{h(t)-1/\alpha(t)} \right) \quad (t \in \mathbb{R}). \quad (5)$$

where $x^{<y>} := \text{sign}(x)|x|^y$, $a : \mathbb{R} \rightarrow \mathbb{R}^+$ is a C^1 function and $\alpha : \mathbb{R} \rightarrow (0, 2)$ and $h : \mathbb{R} \rightarrow (0, 1)$ are C^2 functions.

In fact, lmmm are somewhat more general than said above: indeed, the couple (h, α) allows to prescribe at each point, under certain conditions, both the pointwise Hölder exponent and the local intensity of jumps. In this sense, they generalize both the mBm and the linear multifractional stable motion [75]. From a broader perspective, such multistable multifractional processes are expected to provide relevant models for TCP traces, financial logs, EEG and other phenomena displaying time-varying regularity both in terms of Hölder exponents and discontinuity structure.

Figure 2 displays a graph of an lmmm with linearly increasing α and linearly decreasing H . One sees that the path has large jumps at the beginning, and almost no jumps at the end. Conversely, it is smooth (between jumps) at the beginning, but becomes jaggier and jaggier as time evolves.

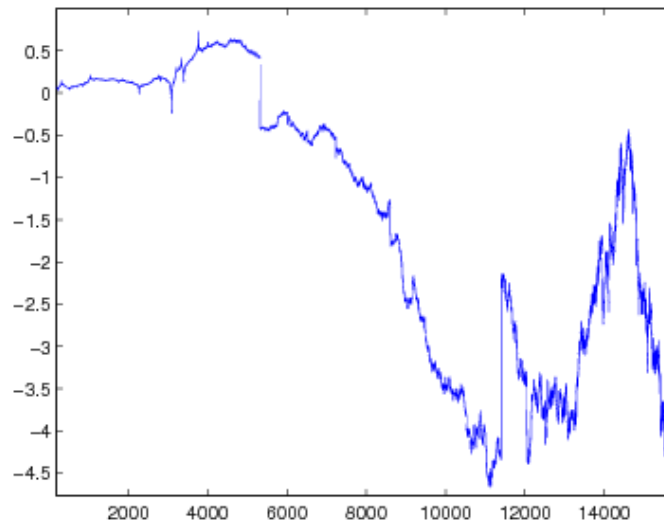


Figure 2. Linear multistable multifractional motion with linearly increasing α and linearly decreasing H

Multiparameter processes

In order to use stochastic processes to represent the variability of multidimensional phenomena, it is necessary to define extensions for indices in \mathbb{R}^N ($N \geq 2$) (see [67] for an introduction to the theory of multiparameter processes). Two different kinds of extensions of multifractional Brownian motion have already been considered: an isotropic extension using the Euclidean norm of \mathbb{R}^N and a tensor product of one-dimensional processes on each axis. We refer to [13] for a comprehensive survey.

These works have highlighted the difficulty of giving satisfactory definitions for increment stationarity, Hölder continuity and covariance structure which are not closely dependent on the structure of \mathbb{R}^N . For example, the Euclidean structure can be unadapted to represent natural phenomena.

A promising improvement in the definition of multiparameter extensions is the concept of *set-indexed processes*. A set-indexed process is a process whose indices are no longer “times” or “locations” but may be some compact connected subsets of a metric measure space. In the simplest case, this framework is a generalization of the classical multiparameter processes [62]: usual multiparameter processes are set-indexed processes where the indexing subsets are simply the rectangles $[0, t]$, with $t \in \mathbb{R}_+^N$.

Set-indexed processes allow for greater flexibility, and should in particular be useful for the modeling of censored data. This situation occurs frequently in biology and medicine, since, for instance, data may not be constantly monitored. Censored data also appear in natural terrain modeling when data are acquired from sensors in presence of hidden areas. In these contexts, set-indexed models should constitute a relevant frame.

A set-indexed extension of fBm is the first step toward the modeling of irregular phenomena within this more general frame. In [18], the so-called *set-indexed fractional Brownian motion (sifBm)* was defined as the mean-zero Gaussian process $\{\mathbf{B}_U^H; U \in \mathcal{A}\}$ such that

$$\forall U, V \in \mathcal{A}; \quad E[\mathbf{B}_U^H \mathbf{B}_V^H] = \frac{1}{2} \left[m(U)^{2H} + m(V)^{2H} - m(U \Delta V)^{2H} \right]$$

where \mathcal{A} is a collection of connected compact subsets of a measure metric space and $0 < H \leq \frac{1}{2}$.

This process appears to be the only set-indexed process whose projection on increasing paths is a one-parameter fractional Brownian motion [17]. The construction also provides a way to define fBm’s extensions on non-euclidean spaces, *e.g.* indices can belong to the unit hyper-sphere of \mathbb{R}^N . The study of fractal properties needs specific definitions for increment stationarity and self-similarity of set-indexed processes [20]. We have proved that the sifBm is the only Gaussian set-indexed process satisfying these two (extended) properties.

In the specific case of the indexing collection $\mathcal{A} = \{[0, t], t \in \mathbb{R}_+^N\} \cup \{\emptyset\}$, the sifBm can be seen as a multiparameter extension of fBm which is called *multiparameter fractional Brownian motion (MpfBm)*. This process differs from the Lévy fractional Brownian motion and the fractional Brownian sheet, which are also multiparameter extensions of fBm (but do not derive from set-indexed processes). The local behaviour of the sample paths of the MpfBm has been studied in [12]. The self-similarity index H is proved to be the almost sure value of the local Hölder exponent at any point, and the Hausdorff dimension of the graph is determined in function of H .

The increment stationarity property for set-indexed processes, previously defined in the study of the sifBm, allows to consider set-indexed processes whose increments are independent and stationary. This generalizes the definition of Bass-Pyke and Adler-Feigin for Lévy processes indexed by subsets of \mathbb{R}^N , to a more general indexing collection. We have obtained a Lévy-Khintchine representation for these set-indexed Lévy processes and we also characterized this class of Markov processes.

4. Application Domains

4.1. Uncertainties management

Our theoretical works are motivated by and find natural applications to real-world problems in a general frame generally referred to as uncertainty management, that we describe now.

Since a few decades, modeling has gained an increasing part in complex systems design in various fields of industry such as automobile, aeronautics, energy, etc. Industrial design involves several levels of modeling: from behavioural models in preliminary design to finite-elements models aiming at representing sharply physical phenomena. Nowadays, the fundamental challenge of numerical simulation is in designing physical systems while saving the experimentation steps.

As an example, at the early stage of conception in aeronautics, numerical simulation aims at exploring the design parameters space and setting the global variables such that target performances are satisfied. This iterative procedure needs fast multiphysical models. These simplified models are usually calibrated using high-fidelity models or experiments. At each of these levels, modeling requires control of uncertainties due to simplifications of models, numerical errors, data imprecisions, variability of surrounding conditions, etc.

One dilemma in the design by numerical simulation is that many crucial choices are made very early, and thus when uncertainties are maximum, and that these choices have a fundamental impact on the final performances.

Classically, coping with this variability is achieved through *model registration* by experimenting and adding fixed *margins* to the model response. In view of technical and economical performance, it appears judicious to replace these fixed margins by a rigorous analysis and control of risk. This may be achieved through a probabilistic approach to uncertainties, that provides decision criteria adapted to the management of unpredictability inherent to design issues.

From the particular case of aircraft design emerge several general aspects of management of uncertainties in simulation. Probabilistic decision criteria, that translate decision making into mathematical/probabilistic terms, require the following three steps to be considered [58]:

1. build a probabilistic description of the fluctuations of the model's parameters (*Quantification of uncertainty sources*),
2. deduce the implication of these distribution laws on the model's response (*Propagation of uncertainties*),
3. and determine the specific influence of each uncertainty source on the model's response variability (*Sensitivity Analysis*).

The previous analysis now constitutes the framework of a general study of uncertainties. It is used in industrial contexts where uncertainties can be represented by *random variables* (unknown temperature of an external surface, physical quantities of a given material, ... at a given *fixed time*). However, in order for the numerical models to describe with high fidelity a phenomenon, the relevant uncertainties must generally depend on time or space variables. Consequently, one has to tackle the following issues:

- *How to capture the distribution law of time (or space) dependent parameters, without directly accessible data?* The distribution of probability of the continuous time (or space) uncertainty sources must describe the links between variations at neighbor times (or points). The local and global regularity are important parameters of these laws, since it describes how the fluctuations at some time (or point) induce fluctuations at close times (or points). The continuous equations representing the studied phenomena should help to *propose models for the law of the random fields*. Let us notice that interactions between various levels of modeling might also be used to derive distributions of probability at the lowest one.
- The navigation between the various natures of models needs a kind of *metric* which could *mathematically describe the notion of granularity or fineness* of the models. Of course, the local regularity will not be totally absent of this mathematical definition.
- All the various levels of conception, preliminary design or high-fidelity modelling, require *registrations by experimentation* to reduce model errors. This *calibration* issue has been present in this frame since a long time, especially in a deterministic optimization context. The random modeling of uncertainty requires the definition of a systematic approach. The difficulty in this specific context is: statistical estimation with few data and estimation of a function with continuous variables using only discrete setting of values.

Moreover, a multi-physical context must be added to these questions. The complex system design is most often located at the interface between several disciplines. In that case, modeling relies on a coupling between several models for the various phenomena and design becomes a *multidisciplinary optimization* problem. In this uncertainty context, the real challenge turns robust optimization to manage technical and economical risks (risk for non-satisfaction of technical specifications, cost control).

We participate in the uncertainties community through several collaborative research projects (ANR and Pôle SYSTEM@TIC), and also through our involvement in the MASCOT-NUM research group (GDR of CNRS). In addition, we are considering probabilistic models as phenomenological models to cope with uncertainties in the DIGITEO ANIFRAC project. As explained above, we focus on essentially irregular phenomena, for which irregularity is a relevant quantity to capture the variability (e.g. certain biomedical signals, terrain modeling, financial data, etc.). These will be modeled through stochastic processes with prescribed regularity.

4.2. Design of complex systems

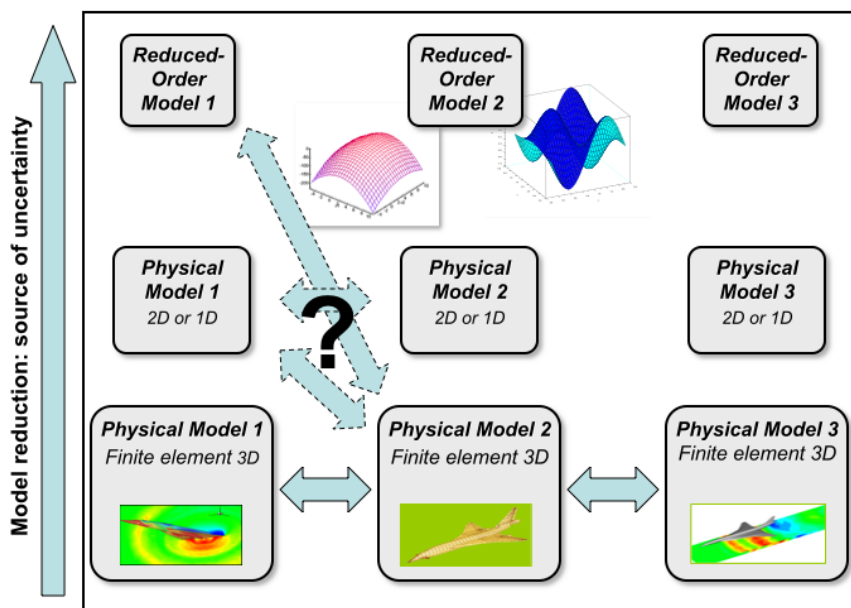


Figure 3. Coupling uncertainty between heterogeneous models

The design of a complex (mechanical) system such as aircraft, automobile or nuclear plant involves numerical simulation of several interacting physical phenomena: CFD and structural dynamics, thermal evolution of a fluid circulation, ... For instance, they can represent the resolution of coupled partial differential equations using finite element method. In the framework of uncertainty treatment, the studied “phenomenological model” is a chaining of different models representing the various involved physical phenomena. As an example, the pressure field on an aircraft wing is the result of both aerodynamic and structural mechanical phenomena. Let us consider the particular case of two models of partial differential equations coupled by limit conditions. The direct propagation of uncertainties is impossible since it requires an exploration and then, many calls to costly models. As a solution, engineers use to build reduced-order models: the complex high-fidelity model is substituted with a CPU less costly model. The uncertainty propagation is then realized through the simplified model, taking into account the approximation error (see [52]).

Interactions between the various models are usually explicitated at the finest level (cf. Fig. 3). How may this coupling be formulated when the fine structures of exchange have disappeared during model reduction? How can be expressed the interactions between models at different levels (in a multi-level modeling)? The ultimate question would be: how to choose the right level of modeling with respect to performance requirements?

In the multi-physical numerical simulation, two kinds of uncertainties then coexist: the uncertainty due to substitution of high-fidelity models with approximated reduced-order models, and the uncertainty due to the new coupling structure between reduced-order models.

According to the previous discussion, the uncertainty treatment in a multi-physical and multi-level modeling implies a large range of issues, for instance numerical resolutions of PDE (which do not enter into the research topics of *Regularity*). Our goal is to contribute to the theoretical arsenal that allows to fly among the different levels of modeling (and then, among the existing numerical simulations). We will focus on the following three axes:

- In the case of a phenomenon represented by two coupled partial differential equations whose resolution is represented by reduced-order models, how to define a probabilistic model of the coupling errors? In connection with our theoretical development, we plan to characterize the regularity of this error in order to quantify its distribution. This research axis is supported by an ANR grant (OPUS project).
- The multi-level modeling assumes the ability to choose the right level of details for the models in adequacy to the goals of the study. In order to do that, a rigorous mathematical definition of the notion of *model fineness/granularity* would be very helpful. Again, a precise analysis of the fine regularity of stochastic models is expected to give elements toward a precise definition of granularity. This research axis is supported by a a Pôle SYSTEM@TIC grant (EHPOC project), and also by a collaboration with EADS.
- Some fine characteristics of the phenomenological model may be used to define the probabilistic behaviour of its variability. The action of modeling a phenomena can be seen as an interpolation issue between given observations. This interpolation can be driven by physical evolution equations or fine analytical description of the physical quantities. We are convinced that Hölder regularity is an essential parameter in that context, since it captures how variations at a given point induce variations at its neighbors. Stochastic processes with prescribed regularity (see section 3.3) have already been used to represent various fluctuating phenomena: Internet traffic, financial data, ocean floor. We believe that these models should be relevant to describe solutions of PDE perturbed by uncertain (random) coefficients or limit conditions. This research axis is supported by a Pôle SYSTEM@TIC grant (CSDL project).

The preliminary design of complex systems can be described as an exploration process of a so-called design space, generated by the global parameters. An interactive exploration, with a decisional visualization goal, needs reduced-order models of the involved physical phenomena. We are convinced that the local regularity of phenomena is a relevant quantity to drive these approximated models. Roughly speaking, in order to be representative, a model needs more informations where the fluctuations are the more important (and consequently, where irregularity is the more important).

In collaboration with Dassault Aviation, EDF and EADS, we study how the local regularity can provide a good quantification of the concept of *granularity* of a model, in order to select the good level of fidelity adapted to the required precision.

Our works in that field can be expressed into:

- The definition and the study of stochastic partial differential equations driven by processes with prescribed regularity (that do not enter into the classical theory of stochastic integration).
- The study of the evolution of the local regularity inside stochastic partial differential equations (SPDE). The stochastic 2-microlocal analysis should provide informations about the local regularity of the solutions, in function of the coefficients of the equations. The knowledge of the fine behaviour of the solution of the SPDE will provide important informations in the view of numerical simulations.

4.3. Biomedical Applications

ECG analysis and modelling

ECG and signals derived from them are an important source of information in the detection of various pathologies, including *e.g.* congestive heart failure, arrhythmia and sleep apnea. The fact that the irregularity of ECG bears some information on the condition of the heart is well documented (see *e.g.* the web resource <http://www.physionet.org>). The regularity parameters that have been studied so far are mainly the box and regularization dimensions, the local Hölder exponent and the multifractal spectrum [61], [63]. These have been found to correlate well with certain pathologies in some situations. From a general point of view, we participate in this research area in two ways.

- First, we use refined regularity characterizations, such as the regularization dimension, 2-microlocal analysis and advanced multifractal spectra for a more precise analysis of ECG data. This requires in particular to test current estimation procedures and to develop new ones.
- Second, we build stochastic processes that mimic in a faithful way some features of the dynamics of ECG. For instance, the local regularity of RR intervals, estimated in a parametric way based on a modelling by an mBm, displays correlations with the amplitude of the signal, a feature that seems to have remained unobserved so far [3]. In other words, RR intervals behave as SRP. We believe that modeling in a simplified way some aspects of the interplay between the sympathetic and parasympathetic systems might lead to an SRP, and to explain both this self-regulating property and the reasons behind the observed multifractality of records. This will open the way to understanding how these properties evolve under abnormal behaviour.

Pharmacodynamics and patient drug compliance

Poor adherence to treatment is a worldwide problem that threatens efficacy of therapy, particularly in the case of chronic diseases. Compliance to pharmacotherapy can range from 5% to 90%. This fact renders clinical tested therapies less effective in ambulatory settings. Increasing the effectiveness of adherence interventions has been placed by the World Health Organization at the top list of the most urgent needs for the health system. A large number of studies have appeared on this new topic in recent years [77], [76]. In collaboration with the pharmacy faculty of Montréal university, we consider the problem of compliance within the context of multiple dosing. Analysis of multiple dosing drug concentrations, with common deterministic models, is usually based on patient full compliance assumption, *i.e.*, drugs are administered at a fixed dosage. However, the drug concentration-time curve is often influenced by the random drug input generated by patient poor adherence behaviour, inducing erratic therapeutic outcomes. Following work already started in Montréal [70], [71], we consider stochastic processes induced by taking into account the random drug intake induced by various compliance patterns. Such studies have been made possible by technological progress, such as the “medication event monitoring system”, which allows to obtain data describing the behaviour of patients.

We use different approaches to study this problem: statistical methods where enough data are available, model-based ones in presence of qualitative description of the patient behaviour. In this latter case, piecewise deterministic Markov processes (PDP) seem a promising path. PDP are non-diffusion processes whose evolution follows a deterministic trajectory governed by a flow between random time instants, where it undergoes a jump according to some probability measure [56]. There is a well-developed theory for PDP, which studies stochastic properties such as extended generator, Dynkin formula, long time behaviour. It is easy to cast a simplified model of non-compliance in terms of PDP. This has allowed us already to obtain certain properties of interest of the random concentration of drug [44]. In the simplest case of a Poisson distribution, we have obtained rather precise results that also point to a surprising connection with infinite Bernoulli convolutions [44], [11], [10]. Statistical aspects remain to be investigated in the general case.

5. Software

5.1. FracLab

Participants: Paul Balança, Jacques Lévy Véhel [correspondant].

FracLab was developed for two main purposes:

1. propose a general platform allowing research teams to avoid the need to re-code basic and advanced techniques in the processing of signals based on (local) regularity.
2. provide state of the art algorithms allowing both to disseminate new methods in this area and to compare results on a common basis.

FracLab is a general purpose signal and image processing toolbox based on fractal, multifractal and local regularity methods. FracLab can be approached from two different perspectives:

- (multi-) fractal and local regularity analysis: A large number of procedures allow to compute various quantities associated with 1D or 2D signals, such as dimensions, Hölder and 2-microlocal exponents or multifractal spectra.
- Signal/Image processing: Alternatively, one can use FracLab directly to perform many basic tasks in signal processing, including estimation, detection, denoising, modeling, segmentation, classification, and synthesis.

A graphical interface makes FracLab easy to use and intuitive. In addition, various wavelet-related tools are available in FracLab.

FracLab is a free software. It mainly consists of routines developed in MatLab or C-code interfaced with MatLab. It runs under Linux, MacOS and Windows environments. In addition, a “stand-alone” version (*i.e.* which does not require MatLab to run) is available.

FracLab has been downloaded several thousands of times in the last years by users all around the world. A few dozens laboratories seem to use it regularly, with more than two hundreds registered users. Our ambition is to make it the standard in fractal softwares for signal and image processing applications. We have signs that this is starting to become the case. To date, its use has been acknowledged in more than two hundreds research papers in various areas such as astrophysics, chemical engineering, financial modeling, fluid dynamics, internet and road traffic analysis, image and signal processing, geophysics, biomedical applications, computer science, as well as in mathematical studies in analysis and statistics (see <http://fraclab.saclay.inria.fr/> for a partial list with papers). In addition, we have opened the development of FracLab so that other teams worldwide may contribute. Additions have been made by groups in Australia, England, France, the USA, and Serbia.

We have produced this year a major release of FracLab (version 2.1).

6. New Results

6.1. A multifractional Hull and White model

Participants: Joachim Lebovits, Jacques Lévy Véhel.

In collaboration with Sylvain Corlay (Paris 6 University).

We have considered the following model, which is an extension of the fractional Hull and White model proposed in [55]: under the risk-neutral measure, the forward price of a risky asset is the solution of the S.D.E.

$$\begin{aligned} dF_t &= F_t \sigma_t dW_t, \\ d \ln(\sigma_t) &= \theta (\mu - \ln(\sigma_t)) dt + \gamma_h d^\circ B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0, \theta > 0, \end{aligned}$$

where B_t^h is a multifractional Brownian motion with regularity function h , and W_t, W_t^σ are standard Brownian motions. This SDE is interpreted in the Wick-Itô sense.

Using functional quantization techniques, it is possible to compute numerically implied forward start volatilities for this model. Using an adequate h function estimated from SP500 data, we have shown that this model is able to reproduce to some extent the volatility surface observed on the market [34].

6.2. Markov characterization of the set-indexed Lévy process

Participant: Erick Herbin.

In collaboration with Prof. Ely Merzbach (Bar Ilan university, Israel).

In [21], the class of set-indexed Lévy processes is considered using the stationarity property defined for the set-indexed fractional Brownian motion in [20]. The general framework of Ivanoff-Merzbach allows to consider standard properties of stochastic processes (e.g. martingale and Markov properties) in the set-indexed context. Processes are indexed by a collection \mathcal{A} of compact subsets of a metric space \mathcal{T} equipped with a Radon measure m , which satisfies several stability conditions. Each process $\{X_U; U \in \mathcal{A}\}$ is assumed to admit an increment process $\{\Delta X_C; C \in \mathcal{C}\}$ defined as an additive extension of X to the collections $\mathcal{C}_0 = \{U \setminus V; U, V \in \mathcal{A}\}$ and

$$\mathcal{C} = \left\{ U \setminus \bigcup_{1 \leq i \leq n} V_i; n \in \mathbf{N}; U, V_1, \dots, V_n \in \mathcal{A} \right\}.$$

A set-indexed process $X = \{X_U; U \in \mathcal{A}\}$ is called a *set-indexed Lévy process* if the following conditions hold

1. $X_{\emptyset'} = 0$ almost surely, where $\emptyset' = \bigcap_{U \in \mathcal{A}} U$.
2. the increments of X are independent: for all pairwise disjoint C_1, \dots, C_n in \mathcal{C} , the random variables $\Delta X_{C_1}, \dots, \Delta X_{C_n}$ are independent.
3. X has m -stationary \mathcal{C}_0 -increments, i.e. for all integer n , all $V \in \mathcal{A}$ and for all increasing sequences $(U_i)_i$ and $(A_i)_i$ in \mathcal{A} , we have

$$[\forall i, m(U_i \setminus V) = m(A_i)] \Rightarrow (\Delta X_{U_1 \setminus V}, \dots, \Delta X_{U_n \setminus V}) \stackrel{(d)}{=} (\Delta X_{A_1}, \dots, \Delta X_{A_n})$$

4. X is continuous in probability: if $(U_n)_{n \in \mathbf{N}}$ is a sequence in \mathcal{A} such that

$$\overline{\bigcup_n \bigcap_{k \geq n} U_k} = \bigcap_n \overline{\bigcup_{k \geq n} U_k} = A \in \mathcal{A}$$

then

$$\lim_{n \rightarrow \infty} P\{|X_{U_n} - X_A| > \epsilon\} = 0$$

On the contrary to previous works of Adler and Feigin (1984) on one hand, and Bass and Pyke (1984) on the other hand, the increment stationarity property allows to obtain explicit expressions for the finite-dimensional distributions of a set-indexed Lévy process. From these, we obtained a complete characterization in terms of Markov properties.

Among the various definitions for Markov property of a SI process, we considered the \mathcal{Q} -Markov property. A collection \mathcal{Q} of functions

$$\begin{aligned} \mathbf{R} \times \mathcal{B}(\mathbf{R}) &\rightarrow \mathbf{R}_+ \\ (x, B) &\mapsto Q_{U,V}(x, B) \end{aligned}$$

where $U, V \in \mathcal{A}(u)$ are s.t. $U \subseteq V$, is called a *transition system* if the following conditions are satisfied:

1. $Q_{U,V}(\bullet, B)$ is a random variable for all $B \in \mathcal{B}(\mathbf{R})$.
2. $Q_{U,V}(x, \bullet)$ is a probability measure for all $x \in \mathbf{R}$.
3. For all $U \in \mathcal{A}(u)$, $x \in \mathbf{R}$ and $B \in \mathcal{B}(\mathbf{R})$, $Q_{U,U}(x, B) = \delta_x(B)$.
4. For all $U \subseteq V \subseteq W \in \mathcal{A}(u)$,

$$\int_{\mathbf{R}} Q_{U,V}(x, dy) Q_{V,W}(y, B) = Q_{U,W}(x, B).$$

A transition system \mathcal{Q} is said

- *spatially homogeneous* if for all $U \subset V$,

$$\forall x \in \mathbf{R}, \forall B \in \mathcal{B}(\mathbf{R}), \quad Q_{U,V}(x, B) = Q_{U,V}(0, B - x);$$

- *m-homogeneous* if $Q_{U,V}$ only depends on $m(V \setminus U)$,
i.e. $\forall U, V, U', V' \in \mathcal{A}(u)$ such that $U \subset V$ and $U' \subset V'$,

$$m(V \setminus U) = m(V' \setminus U') \Rightarrow Q_{U,V} = Q_{U',V'}.$$

A set-indexed process $X := \{X_U; U \in \mathcal{A}\}$ is called *\mathcal{Q} -Markov* if $\forall U, V \in \mathcal{A}(u)$, $U \subseteq V$

$$\forall B \in \mathcal{B}(\mathbf{R}), \quad P[\Delta X_V \in \Gamma \mid \mathcal{F}_U] = Q_{U,V}(\Delta X_U; \Gamma),$$

where $(\mathcal{F}_U)_{U \in \mathcal{A}(u)}$ is the minimal filtration of the process X .

Balan-Ivanoff (2002) proved that any SI process with independent increments is a \mathcal{Q} -Markov process with a spatially homogeneous transition system. The following result proved in [21] shows that the converse is true.

Theorem *Let $X = \{X_U; U \in \mathcal{A}\}$ be a set-indexed process with definite increments. The two following assertions are equivalent:*

1. X is a \mathcal{Q} -Markov process with a spatially homogeneous transition system \mathcal{Q} ;
2. X has independent increments.

This result is strengthened in the following characterization of set-indexed Lévy processes as Markov processes with homogeneous transition systems.

Theorem *Let $X = \{X_U; U \in \mathcal{A}\}$ be a set-indexed process with definite increments and satisfying the stochastic continuity property.*

The two following assertions are equivalent:

1. X is a set-indexed Lévy process ;
2. X is a \mathcal{Q} -Markov process such that $X_\emptyset = 0$ and the transition system \mathcal{Q} is spatially homogeneous and m -homogeneous.

Consequently, if \mathcal{Q} is a transition system which is both spatially homogeneous and m -homogeneous, then there exists a set-indexed process X which is a \mathcal{Q} -Markov process.

6.3. Local Hölder regularity of Set-Indexed processes

Participants: Erick Herbin, Alexandre Richard.

In the set-indexed framework of Ivanoff and Merzbach ([62]), stochastic processes can be indexed not only by \mathbf{R} but by a collection \mathcal{A} of subsets of a measure and metric space (\mathcal{T}, d, m) , with some assumptions on \mathcal{A} . In we introduce and study some assumptions (A_1) and (A_2) on the metric indexing collection $(\mathcal{A}, d_{\mathcal{A}})$ in order to obtain a Kolmogorov criterion for continuous modifications of SI stochastic processes. Under this assumption, the collection is totally bounded and a set-indexed process with good incremental moments will have a modification whose sample paths are almost surely Hölder continuous, for the distance $d_{\mathcal{A}}$. Once this condition is established, we investigate the definition of Hölder coefficients for SI processes. We shall denote $\tilde{\alpha}_X(t)$ and $\alpha_X(t)$ for the local and pointwise Hölder exponents of X at t , and $\tilde{\alpha}_X(t)$ and $\alpha_X(t)$ for their deterministic counterpart in case X is Gaussian.

In [18], a set-indexed extension for fractional Brownian motion has been defined and studied. A mean-zero Gaussian process $\mathbf{B}^H = \{\mathbf{B}_U^H, U \in \mathcal{A}\}$ is called a *set-indexed fractional Brownian motion (SifBm for short)* on $(\mathcal{T}, \mathcal{A}, m)$ if

$$\forall U, V \in \mathcal{A}, \quad \mathbf{E} [\mathbf{B}_U^H \mathbf{B}_V^H] = \frac{1}{2} \left[m(U)^{2H} + m(V)^{2H} - m(U \Delta V)^{2H} \right], \quad (6)$$

where $H \in (0, 1/2]$ is the index of self-similarity of the process.

In [12], $\tilde{\alpha}_X$ and α_X have been determined for the particular case of an SifBm indexed by the collection $\{[0, t]; t \in \mathbf{R}_+^N\} \cup \{\emptyset\}$, called the *multiparameter fractional Brownian motion*. If X denotes the \mathbf{R}_+^N -indexed process defined by $X_t = \mathbf{B}_{[0,t]}^H$ for all $t \in \mathbf{R}_+^N$, it is proved that for all $t_0 \in \mathbf{R}_+^N$, $\tilde{\alpha}_X(t_0) = H$ and with probability one, for all $t_0 \in \mathbf{R}_+^N$, $\alpha_X(t_0) = H$. A theorem of allows one to extend these results to SifBm indexed by a more general class than the sole collection of rectangles of \mathbf{R}_+^N .

Theorem 0.1 Let \mathbf{B}^H be a set-indexed fractional Brownian motion on $(\mathcal{T}, \mathcal{A}, m)$, $H \in (0, 1/2]$. Assume that the subclasses $(\mathcal{A}_n)_{n \in \mathbf{N}}$ satisfy Assumption (A_1) . Then, the local and pointwise Hölder exponents of \mathbf{B}^H at any $U_0 \in \mathcal{A}$, defined with respect to the distance d_m or any equivalent distance, satisfy

$$\mathbf{P} (\forall U_0 \in \mathcal{A}, \tilde{\alpha}_{\mathbf{B}^H}(U_0) = H) = 1$$

and if Assumption (A_2) holds,

$$\mathbf{P} (\forall U_0 \in \mathcal{A}, \alpha_{\mathbf{B}^H}(U_0) = H) = 1.$$

Consequently, since the collection \mathcal{A} of rectangles of \mathbf{R}_+^N with m the Lebesgue measure satisfies (A_1) and (A_2) , we obtained a new result on a classical multiparameter process: the multiparameter fractional Brownian motion \mathbf{B}^H satisfy, for $T \in \mathbf{R}_+^N$:

$$\mathbf{P} (\forall t \in [0, T], \alpha_{\mathbf{B}^H}([0, t]) = \tilde{\alpha}_{\mathbf{B}^H}([0, t]) = H) = 1.$$

6.4. Separability of Set-Indexed Processes

Participant: Alexandre Richard.

A classical result states that any (multiparameter) stochastic process has a separable modification, thus ensuring the measurability property of the sample paths. We extend this result to set-indexed processes.

Let (T, \mathcal{O}) be a topological space. We assume that this space is *second-countable*, ie there exists a countable subset $\tilde{\mathcal{O}} \subseteq \mathcal{O}$ such that any open set of \mathcal{O} can be expressed as a union of elements of $\tilde{\mathcal{O}}$.

A process $\{X_t, t \in T\}$ is *separable* if there exists an at most countable set $S \subset T$ and a null set Λ such that for all closed sets $F \subset \mathbf{R}$ and all open set $O \in \mathcal{O}$,

$$\{\omega : X_s(\omega) \in F \text{ for all } s \in O \cap S\} \setminus \{\omega : X_s(\omega) \in F \text{ for all } s \in O\} \subset \Lambda.$$

This definition is different of the one found in [57], where the space is “linear”, in that this author considers the previous equation only when O is an interval. It happens that this notion needs not be defined in a general topological space. However when restricted to a vector space, our definition implies the previous one.

Theorem 0.2 (Doob’s separability theorem) Any T -indexed stochastic process $X = \{X_t; t \in T\}$ has a separable modification.

If T is an indexing collection in the sense of [62], the topology induced by the distance d_T has to be second-countable. This happens for instance when (T, d_T) is totally bounded, which is the case in

6.5. An increment type set-indexed Markov property

Participant: Paul Balana.

[1] investigates a new approach for the definition of a set-indexed Markov property, named \mathcal{C} -Markov. The study is based on Merzbach and Ivanoff’s set-indexed formalism, i.e. \mathcal{A} denotes a set-indexed collection and \mathcal{C} the family of increments $C = A \setminus B$, where $A \in \mathcal{A}$ and $B \in \mathcal{A}(u)$ (finite unions of sets from \mathcal{A}). Moreover, for any $C = A \setminus B$, $B = \cup_{i=1}^k A_i$, \mathcal{A}_C is defined as the following subset of \mathcal{A} :

$$\mathcal{A}_C = \{U \in \mathcal{A}_\ell; U \not\subseteq B^\circ\} := \{U_C^1, \dots, U_C^p\}, \quad \text{where } p = |\mathcal{A}_C|$$

and \mathcal{A}_ℓ corresponds to the semilattice $\{A_1 \cap \dots \cap A_k, \dots, A_1 \cap A_2, A_1 \dots, A_k\} \subset \mathcal{A}$. The notation \mathbf{X}_C refers to a random vector $\mathbf{X}_C = (X_{U_C^1}, \dots, X_{U_C^p})$. Similarly, \mathbf{x}_C is used to denote a vector of variables $(x_{U_C^1}, \dots, x_{U_C^p})$.

Then, an E -valued set-indexed process $(X_A)_{A \in \mathcal{A}}$ is said to be \mathcal{C} -Markov with respect to a filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ if it is adapted to $(\mathcal{F}_A)_{A \in \mathcal{A}}$ and if it satisfies

$$\mathbb{E}[f(X_A) | \mathcal{G}_C^*] = \mathbb{E}[f(X_A) | \mathbf{X}_C] \quad \mathbb{P}\text{-a.s.} \quad (7)$$

for all $C = A \setminus B \in \mathcal{C}$ and any bounded measurable function $f : E \rightarrow \mathbf{R}$. The sigma-algebra \mathcal{G}_C^* is usually called the strong history of $(\mathcal{F}_A)_{A \in \mathcal{A}}$ and is defined as $\mathcal{G}_C^* = \bigvee_{A \in \mathcal{A}, A \cap C = \emptyset} \mathcal{F}_A$.

The \mathcal{C} -Markov approach has several advantages compared to existing set-indexed Markov literature (mainly \mathcal{Q} -Markov described in [48]). It appears to be a natural extension of the classic one-parameter Markov property. In particular, the concept of transition system can easily extended to our formalism: for any \mathcal{C} -Markov process X , one can defined $\mathcal{P} = \{P_C(\mathbf{x}_C; dx_A); C \in \mathcal{C}\}$ as

$$\forall \mathbf{x}_C \in E^{|\mathcal{A}_C|}, \Gamma \in \mathcal{E}; \quad P_C(\mathbf{x}_C; \Gamma) := \mathbb{P}(X_A \in \Gamma | \mathbf{X}_C = \mathbf{x}_C).$$

A \mathcal{C} -transition system \mathcal{P} happens to satisfy a set-indexed Chapman-Kolmogorov equation,

$$\forall C \in \mathcal{C}, A' \in \mathcal{A}; \quad P_C f = P_{C'} P_{C''} f \quad \text{where } C' = C \cap A', C'' = C \setminus A' \quad (8)$$

and f is a bounded measurable function.

Similarly to the classic Markovian theory, it is proved in [1] that the initial distribution μ and \mathcal{P} characterize entirely the law of a \mathcal{C} -Markov process, and that conversely, for any initial law and any \mathcal{C} -transition system, a corresponding canonical set-indexed \mathcal{C} -Markov process can be constructed. \mathcal{C} -Markov processes enjoy several other properties such as

1. Projections on elementary flows are Markovian;
2. Conditional independence of natural filtrations;
3. Strong Markov property.

The class of set-indexed Lévy processes defined and studied in [21] offers examples of \mathcal{C} -Markov processes whose transition probabilities correspond to

$$\forall C = A \setminus B \in \mathcal{C}, \quad \forall \Gamma \in \mathcal{E}; \quad P_C(\mathbf{x}_C; \Gamma) = \mu^{m(C)}(\Gamma - \Delta x_B), \quad (9)$$

where m is a measure on \mathcal{T} and μ the infinitely divisible probability measure that characterizes the Lévy process. We note that the transition system related the \mathcal{Q} -Markov property has a different form, even if it is related.

Another non-trivial example of \mathcal{C} -Markov process is the set-indexed Ornstein-Uhlenbeck process that has been introduced and studied in [32]. It is a Gaussian Markovian process whose transition densities are given by

$$p_C(\mathbf{x}_C; y) = \frac{1}{\sigma_C \sqrt{2\pi}} \exp \left[-\frac{1}{2\sigma_C^2} \left(y - e^{-\lambda m(A)} \left[\sum_{i=1}^n (-1)^{\varepsilon_i} x_{U_C^i} e^{\lambda m(U_C^i)} \right] \right)^2 \right], \quad (10)$$

where λ and σ are positive parameters, m is a measure on \mathcal{T} and

$$\sigma_C^2 = \frac{\sigma^2}{2\lambda} \left(1 - e^{-2\lambda m(A)} \left[\sum_{i=1}^n (-1)^{\varepsilon_i} e^{2\lambda m(U_C^i)} \right] \right).$$

In the particular case of multiparameter processes, corresponding to the indexing collection $\mathcal{A} = \{[0, t]; t \in \mathbf{R}_+^N\}$, the \mathcal{C} -Markov formalism is related to several existing works. It generalizes the two-parameter $*$ -Markov property introduced in [53] and also embraces the multiparameter Markov property investigated recently in [68]. Finally, under some Feller assumption on the transition system, a multiparameter \mathcal{C} -Markov process is proved to admit a modification with right-continuous sample paths.

6.6. Fine regularity of Lévy processes

Participant: Paul Balança.

This ongoing work focuses on the fine regularity of one-parameter Lévy processes. The main idea of this study is to use the framework of stochastic 2-microlocal analysis (introduced and developed in [16],[33]) to refine sample paths results obtained in [65].

The latter describes entirely the multifractal spectrum of Lévy processes, i.e. the Hausdorff geometry of level sets $(E_h)_{h \in \mathbf{R}_+}$ of the pointwise exponent. These are usually called the *iso-Hölder sets* of X and are given by

$$E_h = \{t \in \mathbf{R} : \alpha_{X,t} = h\} \quad \text{for every } h \in \mathbf{R}_+ \cup \{+\infty\}.$$

The multifractal spectrum is itself defined as the localized Hausdorff dimension of the previous sets, i.e.

$$d_X(h, V) = \dim_{\mathcal{H}}(E_h \cap V) \quad \text{for every } h \in \mathbf{R}_+ \cup \{+\infty\} \text{ and } V \in \mathcal{O} \text{ (open sets in } \mathbf{R}). \quad (11)$$

[65] states that under a mild assumption on the Lévy measure π , a Lévy process X with no Brownian component almost surely satisfies

$$\forall V \in \mathcal{O}; \quad d_X(h, V) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta]; \\ -\infty & \text{if } h \in (1/\beta, +\infty], \end{cases} \quad (12)$$

where the Blumenthal-Gettoor exponent β is given by

$$\beta = \inf \left\{ \delta \geq 0 : \int_{\mathbf{R}^d} (1 \wedge \|x\|^\delta) \pi(dx) < \infty \right\}. \quad (13)$$

Since classic multifractal analysis focuses on the pointwise exponent, it is natural from our point of view to integrate the 2-microlocal frontier into this description. More precisely, we focus on the dichotomy usual/unusual regularity, corresponding to the sets $(\tilde{E}_h)_{h \in \mathbf{R}_+}$ and $(\hat{E}_h)_{h \in \mathbf{R}_+}$:

$$\tilde{E}_h = \{t \in E_h : \forall s' \in \mathbf{R}; \sigma_{X,t}(s') = (h + s') \wedge 0\} \quad \text{and} \quad \hat{E}_h = E_h \setminus \tilde{E}_h.$$

The collection $(\tilde{E}_h)_{h \in \mathbf{R}_+}$ represents times at which the 2-microlocal behaviour is rather common (i.e. the slope is equal to one), whereas at points which belong $(\hat{E}_h)_{h \in \mathbf{R}_+}$, the 2-microlocal frontier has an unusual form.

Then, our main result states that sample paths of a Lévy process X with no Brownian component almost surely satisfy

$$\forall V \in \mathcal{O}; \quad \dim_{\mathcal{H}}(\tilde{E}_h \cap V) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta]; \\ -\infty & \text{if } h \in (1/\beta, +\infty]. \end{cases} \quad (14)$$

Furthermore, the collection of sets $(\hat{E}_h)_{h \in \mathbf{R}_+}$ enjoys almost surely

$$\forall V \in \mathcal{O}; \quad \dim_{\mathcal{H}}(\hat{E}_h \cap V) \leq \begin{cases} 2\beta h - 1 & \text{if } h \in (1/2\beta, 1/\beta); \\ -\infty & \text{if } h \in [0, 1/2\beta] \cup [1/\beta, +\infty]. \end{cases} \quad (15)$$

These results clearly extend those obtained in [65] since we know that the pointwise exponent is completely characterize by the 2-microlocal frontier. Moreover, it also proves that from a Hausdorff dimension point of view, the common regularity is a 2-microlocal frontier with a slope equal to one.

Nevertheless, equation (15) also exhibits some unusual behaviours, corresponding to times $(\hat{E}_h)_{h \in \mathbf{R}_+}$, that are not captured by the classic multifractal spectrum. The existence of such particular times highly depends on the structure of the Lévy measure, and not only the value of the Blumenthal-Gettoor exponent which is therefore not sufficient to characterize entirely the fine regularity. This last aspect of the study illustrates the fact that 2-microlocal analysis is an interesting tool for the study of stochastic processes' regularity since some sample paths' properties can not be captured by common tools such as Hölder exponents.

6.7. A class of self-similar processes with stationary increments in higher order Wiener chaoses.

Participant: Benjamin Arras.

Self similar processes with stationary increments (SSSI processes) have been studied for a long time due to their importance both in theory and in practice. Such processes appear as limits in various renormalisation procedures [69]. In applications, they occur in various fields such as hydrology, biomedicine and image processing. The simplest SSSI processes are simply Brownian motion and, more generally, Lévy stable motions. Apart from these cases, the best known such process is probably fractional Brownian motion (fBm). A construction of SSSI processes that generalizes fBm to higher order Wiener chaoses was proposed in [73]. These processes read

$$\forall t \in \mathbb{R}_+ \quad X_t = \int_{\mathbb{R}^d} h_t^H(x_1, \dots, x_d) dB_{x_1} \dots dB_{x_d}$$

where h_t^H verifies:

1. $h_t^H \in \widehat{L}^2(\mathbb{R}^d)$,
2. $\forall c > 0, \quad h_{ct}^H(cx_1, \dots, cx_d) = c^{H-\frac{d}{2}} h_t^H(x_1, \dots, x_d)$,
3. $\forall \rho \geq 0, \quad h_{t+\rho}^H(x_1, \dots, x_d) - h_t^H(x_1, \dots, x_d) = h_\rho^H(x_1 - t, \dots, x_d - t)$.

In [41], we define a class of such processes by the following multiple Wiener-Itô integral representation:

$$X_t^\alpha = \int_{\mathbb{R}^d} \left[\|\mathbf{t}^* - \mathbf{x}\|_2^{H-\frac{d}{2}} - \|\mathbf{x}\|_2^{H-\frac{d}{2}} \right] dB_{x_1} \dots dB_{x_d} \quad (16)$$

where $t \in [0, 1]$, $\mathbf{t}^* = (t, \dots, t)$ and $\alpha = H - 1 + \frac{d}{2}$. When $d = 1$, this is just fBm. In order to study the local regularity of this class of processes as well as the asymptotic behaviour at infinity, we use wavelet's methods. More precisely, following ideas from [46], we obtain the following wavelet-like expansion: Almost surely,

$$\forall t \in [0, 1] \quad X_t^\alpha = \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{\epsilon \in E} 2^{-jH} \left[I^{\alpha+1}(\psi^{(\epsilon)})(2^j \mathbf{t}^* - \mathbf{k}) - I^{\alpha+1}(\psi^{(\epsilon)})(-\mathbf{k}) \right] I_d(\psi_{j, \mathbf{k}}^{(\epsilon)}).$$

From this representation, we get several results about this class of processes. Namely:

- There exists a strictly positive random variable A_d of finite moments of any order and a constant, $b_d > 1$, such that:

$$\forall \omega \in \Omega^* \quad \sup_{(s,t) \in [0,1]} \frac{|X_t^\alpha(\omega) - X_s^\alpha(\omega)|}{|t-s|^H (\log(b_d + |t-s|^{-1}))^{\frac{d}{2}}} \leq A_d(\omega).$$

- There exists a strictly positive random variable B_d of finite moments of any order and a constant $c_d > 3$, such that:

$$\forall \omega \in \Omega^* \quad \sup_{t \in \mathbb{R}_+} \frac{|X_t^\alpha(\omega)|}{(1+|t|)^H (\log \log(c_d + |t|))^{\frac{d}{2}}} \leq B_d(\omega).$$

Using an estimate from [54], we compute the uniform almost sure pointwise Hölder exponent of X^α defined by:

$$\gamma_{X^\alpha}(t) = \sup \{ \gamma > 0 : \limsup_{\rho \rightarrow 0} \frac{|X_{t+\rho}^\alpha - X_t^\alpha|}{|\rho|^\gamma} < +\infty \}.$$

We get the following result:

Almost surely,

$$\forall t \in (0, 1), \gamma_{X^\alpha}(t) = H.$$

In the last part of [41], we give general bounds on the Hausdorff dimension of the range and graphs of multidimensional anisotropic SSSI processes defined by multiple Wiener integrals. Let $Y_t^H = \gamma(H, d)I_d(h_t^H)$ where $\gamma(H, d)$ is a normalizing positive constant such that $\mathbb{E}[|Y_1^H|^2] = 1$. Let $\frac{1}{2} < H_1 \leq \dots \leq H_N < 1$. Let $\{\mathbb{Y}_t^H\}$ be the multidimensional process defined by:

$$\{\mathbb{Y}_t^H\} = \{(Y_t^{H_1}, \dots, Y_t^{H_N}) : t \in \mathbb{R}_+\}$$

where the coordinates are independent copies of the process Y_t^H . Following classical ideas from [78] and using again the estimate from [54], we obtain:

Almost surely,

$$\dim_{\mathcal{H}} R_E(\mathbb{Y}^H) \geq \min \left(N; \frac{\dim_{\mathcal{H}} E + \frac{\sum_{j=1}^k (H_k - H_j)}{d}}{H_k}, k = 1, \dots, N \right),$$

$$\dim_{\mathcal{H}} Gr_E(\mathbb{Y}^H) \geq \min \left(\frac{\dim_{\mathcal{H}} E + \frac{\sum_{j=1}^k (H_k - H_j)}{d}}{H_k}, k = 1, \dots, N, \dim_{\mathcal{H}} E + \sum_{i=1}^N \frac{(1 - H_i)}{d} \right).$$

And,

$$\dim_{\mathcal{H}} R_E(\mathbb{Y}^H) \leq \min \left(N; \frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N \right),$$

$$\dim_{\mathcal{H}} Gr_E(\mathbb{Y}^H) \leq \min \left(\frac{\dim_{\mathcal{H}} E + \sum_{j=1}^k (H_k - H_j)}{H_k}, k = 1, \dots, N; \dim_{\mathcal{H}} E + \sum_{i=1}^N (1 - H_i) \right).$$

where $E \subset \mathbb{R}_+$.

6.8. Economic growth models

Participants: Jacques Lévy Véhel, Lining Liu.

In collaboration with D. La Torre, University of Milan.

We study certain economic growth models where we add a source of randomness to make the evolution equations more realistic. We have studied two particular models:

- An augmented Uzawa-Lucas growth model where technological progress is modelled as the solution of a stochastic differential equation driven by a Lévy or an additive process. This allows for a more faithful description of reality by taking into account discontinuities in the evolution of the level of technology. In details, we consider a closed economy in which there is single good which is produced by combining physical capital $K(t)$ and human capital $H(t)$. The laws of motions of $K(t)$ and $H(t)$ are:

$$\dot{K}(t) = A(t)^\gamma [u(t)H(t)]^\xi K(t)^{1-\xi-\gamma} - \beta_K K(t) - C(t), \quad (17)$$

$$K(0) = K_0;$$

$$\dot{H}(t) = (\eta(1 - u(t)) - \beta_H)H(t), \quad (18)$$

$$H(0) = H_0,$$

where $A(t)$ is the level of technology, $H(t)$ is the total stock of human capital, $u(t)$ is the proportion to the production of good, $\gamma \in (0, 1)$, $\xi \in (0, 1)$ and $1 - \xi - \gamma \in (0, 1)$ are the shares of income accruing to $A(t)$, $u(t)H(t)$ and $K(t)$, respectively, $\beta_K \in [0, 1]$ is the constant rate of depreciation of physical capital, $\beta_H \in [0, 1]$ is the rate of depreciation of human capital and $\eta \geq 0$ is the productivity of human capital.

We assume that the level of technology evolves according to the following stochastic differential equation:

$$dA(t) = \mu A(t)dt + \sigma A(t)dW(t) + \delta \int A(t^-)z(\tilde{N}(dt, dz) - \nu(dt, dz)), \quad (19)$$

where $\mu \in \mathbb{R}$ is the drift rate, $\sigma > 0$ is the volatility, $0 \leq \delta \leq 1$, W is a standard Brownian motion and \tilde{N} is Poisson random measure with intensity measure ν which satisfies

$$\lim_{s \rightarrow t^+} \frac{1}{s-t} \int_t^s \int_{-1}^1 z^2 \nu(dz, dx) + \lim_{s \rightarrow t^+} \frac{1}{s-t} \int_t^s \int_1^\infty z \nu(dz, dx) < \infty,$$

and

$$\int_0^t \int_{-1}^1 z^2 \nu(dz, dx) + \int_0^t \int_1^\infty z \nu(dz, dx) < \infty,$$

for $t > 0$.

With a CIES utility function, the optional inter-temporal decision problem can be formulated as

$$\max_{[C, u]} \mathbb{E} \left[\int_0^\infty \frac{C(t)^{1-\phi} - 1}{1-\phi} e^{-\rho t} dt \right], \quad (20)$$

where $\rho > 0$ is the rate of time preference and $\phi > 0$. We denote $V(H, K, A)$ the maximum value function associated with the stochastic optimisation problem. For given t , the maximum expect utility up to time t obtained when applying the stochastic control $[C(t), u(t)]$ is defined by

$$V(H(t), K(t), A(t)) = \max_{[C, u]} \mathbb{E} \left[\int_0^t \frac{C(x)^{1-\phi} - 1}{1-\phi} e^{-\rho x} dx \right]. \quad (21)$$

We have been able to solve this program under some simplifying assumptions. Numerical simulations allow one to assess precisely the effect of (tempered) multistable noise on the model.

- A stochastic demographic jump shocks in a multi-sector growth model with physical and human capital accumulation. This models allows one to take into account sudden changes in population size, due for instance to wars or natural catastrophes. The laws of motions of physical capital $K(t)$ and human capital $H(t)$ are:

$$\dot{K}(t) = AM(t)^{1-\xi-\beta}[u(t)H(t)]^\beta K(t)^\xi - \eta_K K(t) - c(t)M(t), \quad (22)$$

$$\dot{H}(t) = B(1 - u(t))H(t) - \eta_H H(t), \quad (23)$$

with initial conditions $K(0) = K_0$ and $H(0) = H_0$, where $M(t)$ is the population size, $H(t)$ is the human capital, $u(t)$ is the share of human capital employed in production, $\beta \in (0, 1)$, $\xi \in (0, 1)$ and $1 - \xi - \beta \in (0, 1)$ are the shares accruing to $M(t)$, $u(t)H(t)$ and $K(t)$, respectively, $\eta_K \in [0, 1]$ is the constant rate of depreciation of physical capital, $\eta_H \in [0, 1]$ is the rate of depreciation of human capital and $A \geq 0$, $B \geq 0$ are the productivities of physical capital and human capital.

We assume that the population size evolves according to the following stochastic differential equation:

$$dM(t) = \mu M(t)dt + \sigma M(t)dW(t) + \delta \int M(t^-)z(\tilde{N}(dt, dz) - \nu(dt, dz))$$

with initial condition $M(0) = M_0$, where $\mu \in \mathbb{R}$ is the drift rate, $\sigma > 0$ is the volatility, $0 \leq \delta \leq 1$, W is a standard Brownian motion and \tilde{N} is Poisson random measure with intensity measure $\nu(dt, dz)$.

Here again, we are able to solve an optimisation program under some simplifying assumptions. This sheds light on the effect of demographic shocks on macroeconomic growth.

7. Bilateral Contracts and Grants with Industry

7.1. Bilateral Contracts with Industry

CSDL (Complex Systems Design Lab) project of the Pôle de Compétitivité SYSTEM@TIC PARIS-REGION (11/2009-10/2012). The goal of the project is the development of a scientific platform of decisional visualization for preliminary design of complex system. Industrial partners include Dassault Aviation, EADS, EDF, MBDA and Renault. Academic partners include ECP, Ecole des Mines de Paris, ENS Cachan, Inria and Supelec.

8. Partnerships and Cooperations

8.1. National Initiatives

Erick Herbin is member of the CNRS Research Groups:

- GDR Mascot Num, devoted to stochastic analysis methods for codes and numerical treatment;
- GDR Math-Entreprise, devoted to mathematical modeling of industrial issues.

8.2. International Initiatives

8.2.1. Inria International Partners

- Regularity collaborates with Bar Ilan university on theoretical developments around set-indexed fractional Brownian motion and set-indexed Lévy processes (invitations of Erick Herbin in Israël during five months in 2006, 2007, 2008, 2009 and 2011 and invitation of Prof. Ely Merzbach at Ecole Centrale Paris in 2008, 2009, 2010 and 2011). The PhD thesis of Alexandre Richard is co-supervised by Erick Herbin and Ely Merzbach.

Erick Herbin was invited to the Mathematics Colloquium (Bar Ilan University, Israel) in July, 2012. Talk: "Hausdorff dimension of the graph of Gaussian processes".

- Regularity collaborates with Michigan State University (Prof. Yimin Xiao) on the study of fine regularity of multiparameter fractional Brownian motion (invitation of Erick Herbin at East Lansing in 2010).
- Regularity collaborates with St Andrews University (Prof. Kenneth Falconer) on the study of multistable processes.
- Regularity collaborates with Acadia University (Prof. Franklin Mendivil) on the study of fractal strings, certain fractals sets, and the study of the regularization dimension.
- Regularity collaborates with Milan University (Prof. Davide La Torre) on the study of certain economic growth models. A joint project has just been selected in the frame of the Galilée program.

8.3. International Research Visitors

8.3.1. Visits of International Scientists

Professors Ely Merzbach from Bar Ilan University and Franklin Mendivil from Acadia University have visited the team this year.

8.3.1.1. Internships

Ankush GOYAL (from May 2012 until Jul 2012)

Subject: Stochastic calculus with multistable Lévy motion and applications in finance

Institution: IIT Delhi (India)

9. Dissemination

9.1. Scientific Animation

Jacques Lévy Véhel is associate editor of the journal *Fractals*.

Jacques Lévy Véhel was invited for two weeks at the program *Stochastic Analysis* organized by the Bernoulli Center, Lausanne.

9.2. Teaching - Supervision - Juries

9.2.1. Teaching

- Erick Herbin is head of the Mathematics Department at Ecole Centrale Paris since 2011.
- Erick Herbin is in charge of the "Mathematical Modeling and Numerical Simulation" Program in the Applied Mathematics option of Ecole Centrale Paris.
- Erick Herbin is in charge of the Probability course at Ecole Centrale Paris (20h).
- Erick Herbin is in charge of the Advanced Probability course at Ecole Centrale Paris (30h).

- Erick Herbin and Jacques Lévy Véhel are in charge of the Brownian Motion and Stochastic Calculus course at Ecole Centrale Paris (30h).
- Erick Herbin gives tutorials on Real Analysis and Integration at Ecole Centrale Paris (10h).
- Jacques Lévy Véhel teaches a course on Wavelets and Fractals at Ecole Centrale Nantes (8h).
- Paul Balança and Alexandre Richard are teaching assistants since October 2010 at Ecole Centrale Paris:
 - Paul Balança gives tutorials on Probability, Real Analysis and Integration at Ecole Centrale Paris (20h).
 - Alexandre Richard gives tutorials on Probability and Statistics at Ecole Centrale Paris (20h).
 - Paul Balança and Alexandre Richard give tutorials on Advanced Probability at Ecole Centrale Paris (17h).
- Paul Balança, Erick Herbin and Alexandre Richard supervise several student's research projects in the field of Mathematics at Ecole Centrale Paris.

9.2.2. Supervision

PhD : Joachim Lebovits, Stochastic Calculus With Respect to Multifractional Brownian Motion and Applications to Finance, Université de Paris 6, defended on January 25, 2012, supervised by J. Lévy Véhel and M. Yor.

PhD in progress : Benjamin Arras, Self-similar processes in higher order chaoses, started in September 2011, supervised by J. Lévy Véhel.

PhD in progress : Paul Balança, Stochastic 2-microlocal analysis of SDEs, started in October 2010, supervised by Erick Herbin.

PhD in progress : Alexandre Richard, Regularity of set-indexed processes and construction of a set-indexed process with varying local regularity , started in October 2010, supervised by Erick Herbin and E. Merzbach.

9.3. Popularization

J. Lévy Véhel has written articles for Interstices and the "le saviez-vous" page of Inria web site.

10. Bibliography

Major publications by the team in recent years

- [1] P. BALANÇA. *A increment type set-indexed Markov property*, 2012, <http://arxiv.org/abs/1207.6568>.
- [2] J. BARRAL, J. LÉVY-VÉHEL. *Multifractal Analysis of a Class of Additive Processes with Correlated Non-Stationary Increments*, in "Electronic Journal of Probability", 2004, vol. 9, p. 508–543.
- [3] O. BARRIÈRE, J. LÉVY-VÉHEL. *Application of the Self Regulating Multifractional Process to cardiac interbeats intervals*, in "J. Soc. Fr. Stat.", 2009, vol. 150, n^o 1, p. 54–72.
- [4] F. CHALOT, Q. V. DINH, E. HERBIN, L. MARTIN, M. RAVACHOL, G. ROGÉ. *Estimation of the impact of geometrical uncertainties on aerodynamic coefficients using CFD*, in "10th AIAA Non-Deterministic Approaches", Schaumburg, USA, April 2008.

- [5] K. DAOUDI, J. LÉVY-VÉHEL, Y. MEYER. *Construction of continuous functions with prescribed local regularity*, in "Journal of Constructive Approximation", 1998, vol. 014, n° 03, p. 349–385.
- [6] Y. DEREMAUX, J. NÉGRIER, N. PIÉTREMONT, E. HERBIN, M. RAVACHOL. *Environmental MDO and uncertainty hybrid approach applied to a supersonic business jet*, in "12th AIAA/ISSMO Multidisciplinary Analysis and Optimization conference", 2008, Victoria.
- [7] A. ECHELARD, O. BARRIÈRE, J. LÉVY-VÉHEL. *Terrain modelling with multifractional Brownian motion and self-regulating processes*, in "ICCVG 2010", Warsaw, Poland, Lecture Notes in Computer Science, Springer, 2010, vol. 6374, p. 342-351, <http://hal.inria.fr/inria-00538907/en>.
- [8] K. FALCONER, R. LE GUÉVEL, J. LÉVY-VÉHEL. *Localisable moving average stable and multistable processes*, in "Stoch. Models", 2009, vol. 25, p. 648–672.
- [9] K. FALCONER, J. LÉVY-VÉHEL. *Multifractional, multistable, and other processes with prescribed local form*, in "J. Theoret. Probab.", 2008, vol. 119, p. 2277–2311, DOI 10.1007/s10959-008-0147-9.
- [10] L. FERMIN, J. LÉVY-VÉHEL. *Modeling patient poor compliance in in the multi-IV administration case with Piecewise Deterministic Markov Models*, 2011, preprint.
- [11] L. FERMIN, J. LÉVY-VÉHEL. *Variability and singularity arising from poor compliance in a pharmacodynamical model II: the multi-oral case*, 2011, preprint.
- [12] E. HERBIN, B. ARRAS, G. BARRUEL. *From almost sure local regularity to almost sure Hausdorff dimension for Gaussian fields*, 2010, preprint.
- [13] E. HERBIN. *From n parameter fractional brownian motions to n parameter multifractional brownian motions*, in "Rocky Mountain Journal of Mathematics", 2006, vol. 36, n° 4, p. 1249–1284.
- [14] E. HERBIN, J. JAKUBOWSKI, M. RAVACHOL, Q. V. DINH. *Management of uncertainties at the level of global design*, in "Symposium "Computational Uncertainties", RTO AVT-147", 2007, Athens.
- [15] E. HERBIN, J. LÉBOVITS, J. LÉVY-VÉHEL. *Stochastic integration with respect to multifractional Brownian motion via tangent fractional Brownian motion*, in "preprint", 2011.
- [16] E. HERBIN, J. LÉVY-VÉHEL. *Stochastic 2-microlocal analysis*, in "Stochastic Proc. Appl.", 2009, vol. 119, n° 7, p. 2277–2311, <http://arxiv.org/abs/math.PR/0504551>.
- [17] E. HERBIN, E. MERZBACH. *A characterization of the set-indexed fractional Brownian motion*, in "C. R. Acad. Sci. Paris", 2006, vol. Ser. I 343, p. 767–772.
- [18] E. HERBIN, E. MERZBACH. *A set-indexed fractional brownian motion*, in "J. of theor. probab.", 2006, vol. 19, n° 2, p. 337–364.
- [19] E. HERBIN, E. MERZBACH. *The multiparameter fractional Brownian motion*, in "Math everywhere", Berlin, Springer, Berlin, 2007, p. 93–101, http://dx.doi.org/10.1007/978-3-540-44446-6_8.

- [20] E. HERBIN, E. MERZBACH. *Stationarity and self-similarity characterization of the set-indexed fractional Brownian motion*, in "J. of theor. probab.", 2009, vol. 22, n^o 4, p. 1010–1029.
- [21] E. HERBIN, E. MERZBACH. *The set-indexed Lévy process: Stationarity, Markov and sample paths properties*, 2010, preprint.
- [22] E. HERBIN, A. RICHARD. *Hölder regularity for set-indexed processes*, in "Submitted", 2011, submitted.
- [23] K. KOLWANKAR, J. LÉVY-VÉHEL. *A time domain characterization of the fine local regularity of functions*, in "J. Fourier Anal. Appl.", 2002, vol. 8, n^o 4, p. 319–334.
- [24] J. LÉBOVITS, J. LÉVY-VÉHEL. *Stochastic Calculus with respect to multifractional Brownian motion*, submitted, <http://hal.inria.fr/inria-00580196/en>.
- [25] J. LÉVY-VÉHEL, M. RAMS. *Large Deviation Multifractal Analysis of a Class of Additive Processes with Correlated Non-Stationary Increments*, submitted, <http://hal.inria.fr/inria-00633195/en>.
- [26] J. LÉVY-VÉHEL, C. TRICOT. *On various multifractal spectra*, in "Fractal Geometry and Stochastics III, Progress in Probability", Birkhäuser, ISBN 376437070X, 9783764370701, 2004, vol. 57, p. 23-42, C. Bandt, U. Mosco and M. Zähle (Eds), Birkhäuser Verlag.
- [27] J. LÉVY-VÉHEL, R. VOJAK. *Multifractal Analysis of Choquet Capacities: Preliminary Results*, in "Advances in Applied Mathematics", January 1998, vol. 20, p. 1–43.
- [28] R. PELTIER, J. LÉVY-VÉHEL. *Multifractional Brownian Motion*, Inria, 1995, n^o 2645, <http://hal.inria.fr/inria-00074045>.
- [29] M. RAVACHOL, Y. DEREMAUX, Q. V. DINH, E. HERBIN. *Uncertainties at the conceptual stage: Multilevel multidisciplinary design and optimization approach*, in "26th International Congress of the Aeronautical Sciences", 2008, Anchorage.
- [30] F. ROUEFF, J. LÉVY-VÉHEL. *A Regularization Approach to Fractional Dimension Estimation*, in "Fractals'98", 1998, Malta.
- [31] S. SEURET, J. LÉVY-VÉHEL. *A time domain characterization of of 2-microlocal Spaces*, in "J. Fourier Anal. Appl.", 2003, vol. 9, n^o 5, p. 472–495.

Publications of the year

Articles in International Peer-Reviewed Journals

- [32] P. BALANÇA, E. HERBIN. *A set-indexed Ornstein-Uhlenbeck process*, in "Electronic Communications in Probability", 2012, vol. 17, n^o 39, p. 1-14, 13 pages [DOI : 10.1214/ECP.v17-1903], <http://hal.inria.fr/hal-00734421>.
- [33] P. BALANÇA, E. HERBIN. *2-microlocal analysis of martingales and stochastic integrals*, in "Stochastic Processes and their Applications", 2012, vol. 122, p. 2346-2382, 40 pages, 3 figures [DOI : 10.1016/J.SPA.2012.03.011], <http://hal.inria.fr/hal-00734418>.

- [34] S. CORLAY, J. LEBOVITS, J. LÉVY-VÉHEL. *Multifractional Stochastic volatility models*, in "Mathematical Finance", September 2012, Accepted for Publication, <http://hal.inria.fr/hal-00653150>.
- [35] R. LE GUÉVEL, J. LÉVY-VÉHEL. *A Ferguson - Klass - LePage series representation of multistable multifractional processes and related processes*, in "Bernoulli", 2012, vol. 18, n° 4, p. 1099-1127 [DOI : 10.3150/11-BEJ372], <http://hal.inria.fr/inria-00538985>.
- [36] J. LÉVY-VÉHEL, F. MENDIVIL. *Local complex dimensions of a fractal string*, in "International Journal of mathematical modelling and numerical optimisation", October 2012, vol. 3, n° 4, <http://hal.inria.fr/inria-00614665>.
- [37] L. TRUJILLO, P. LEGRAND, G. OLAGUE, J. LÉVY-VÉHEL. *Evolving Estimators of the Pointwise Holder Exponent with Genetic Programming*, in "Information Sciences", 2012, vol. 209, p. 61-79, Submitted [DOI : 10.1016/J.INS.2012.04.043], <http://hal.inria.fr/hal-00643387>.

International Conferences with Proceedings

- [38] A. ECHELARD, J. LÉVY-VÉHEL. *Self-regulating processes-based modelling for arrhythmia characterization*, in "Imaging and Signal Processing in Health Care and Technology", Baltimore, United States, May 2012, <http://hal.inria.fr/hal-00670064>.

- [39] *Best Paper*
J. LÉVY-VÉHEL, M. TESMER. *A New Method for Multifractal Spectrum Estimation with Applications to Texture Description*, in "International Conference on Mass Data Analysis of Images and Signals (MDA'2012)", Berlin, Germany, July 2012, <http://hal.inria.fr/hal-00686408>.

Conferences without Proceedings

- [40] A. ECHELARD, J. LÉVY-VÉHEL, C. TRICOT. *A Unified Framework for the Study of the 2-microlocal and Large Deviation Multifractal Spectra*, in "Self similar processes and their applications", Angers, France, SMF, 2012, p. 13-44, <http://hal.inria.fr/inria-00612342>.

Other Publications

- [41] B. ARRAS. *On a class of self-similar processes with stationary increments in higher order Wiener chaoses*, 2012, 21 pages, <http://hal.inria.fr/hal-00759165>.
- [42] O. BARRIÈRE, A. ECHELARD, J. LÉVY-VÉHEL. *Self-Regulating Processes*, To appear in the Electronic Journal of Probability, <http://hal.inria.fr/hal-00749742>.
- [43] R. LE GUÉVEL, J. LÉVY-VÉHEL, L. LIU. *On two multistable extensions of stable Lévy motion and their semimartingale representation*, <http://hal.inria.fr/hal-00730680>.
- [44] P.-E. LÉVY-VÉHEL, J. LÉVY-VÉHEL. *Variability and singularity arising from poor compliance in a pharmacokinetic model I: the multi-IV case*, To appear in Journal of Pharmacokinetics and Pharmacodynamics., <http://hal.inria.fr/hal-00752114>.

- [45] J. LÉVY-VÉHEL, F. MENDIVIL. *Christiane's Hair*, To appear in the American Mathematical Monthly, <http://hal.inria.fr/hal-00744268>.

References in notes

- [46] A. AYACHE, Y. XIAO. *Asymptotic properties and Hausdorff dimensions of fractional Brownian sheets*, in "J. Fourier Anal. Appl.", 2005, vol. 11, n^o 4, p. 407–439, <http://dx.doi.org/10.1007/s00041-005-4048-3>.
- [47] F. BACCELLI, D. HONG. *AIMD, Fairness and Fractal Scaling of TCP Traffic*, in "INFOCOM'02", June 2002.
- [48] R. M. BALAN, G. IVANOFF. *A Markov property for set-indexed processes*, in "J. Theoret. Probab.", 2002, vol. 15, n^o 3, p. 553–588, <http://dx.doi.org/10.1023/A:1016296330187>.
- [49] A. BENASSI, S. JAFFARD, D. ROUX. *Elliptic Gaussian random processes*, in "Rev. Mathématique Iberoamericana", 1997, vol. 13, n^o 1, p. 19–90.
- [50] J. BONY. *Second microlocalization and propagation of singularities for semilinear hyperbolic equations*, in "Conf. on Hyperbolic Equations and Related Topics", 1984, p. 11–49, Kata/Kyoto, Academic Press, Boston.
- [51] G. BROWN, G. MICHON, J. PEYRIÈRE. *On the multifractal analysis of measures*, in "J. Statist. Phys.", 1992, vol. 66, n^o 3, p. 775–790.
- [52] D. CACUCI. *Sensitivity and Uncertainty Analysis, Volume 1: Theory.*, Chapman & Hall/CRC, 2003.
- [53] R. CAIROLI. *Une classe de processus de Markov*, in "C. R. Acad. Sci. Paris Sér. A-B", 1971, vol. 273, p. A1071–A1074.
- [54] A. CARBERY, J. WRIGHT. *Distributional and L^q norm inequalities for polynomials over convex bodies in R^n* , in "Math. Res. Lett.", 2001, vol. 8, n^o 3, p. 233–248.
- [55] F. COMTE, E. RENAULT. *Long memory in continuous-time stochastic volatility models*, in "Mathematical Finance", 1998, vol. 8, n^o 4, p. 291–323, <http://dx.doi.org/10.1111/1467-9965.00057>.
- [56] M. DAVIS. *Markov Models and Optimization*, Chapman & Hall, London, 1993.
- [57] J. DOOB. *Stochastic Processes*, Wiley, 1953.
- [58] ESREDA. *Uncertainty in Industrial Practice, a Guide to Quantitative Uncertainty Management*, Wiley, 2009.
- [59] K. FALCONER. *The local structure of random processes*, in "J. London Math. Soc.", 2003, vol. 2, n^o 67, p. 657–672.
- [60] K. FALCONER. *The multifractal spectrum of statistically self-similar measures*, in "J. Theor. Prob.", 1994, vol. 7, p. 681–702.
- [61] A. GOLDBERGER, L. A. N. AMARAL, J. HAUSDORFF, P. IVANOV, C. PENG, H. STANLEY. *Fractal dynamics in physiology: Alterations with disease and aging*, in "PNAS", 2002, vol. 99, p. 2466–2472.

- [62] G. IVANOFF, E. MERZBACH. *Set-Indexed Martingales*, Chapman & Hall/CRC, 2000.
- [63] P. IVANOV, L. A. N. AMARAL, A. GOLDBERGER, S. HAVLIN, M. ROSENBLUM, Z. STRUZIK, H. STANLEY. *Multifractality in human heartbeat dynamics*, in "Nature", June 1999, vol. 399.
- [64] S. JAFFARD. *Pointwise smoothness, two-microlocalization and wavelet coefficients*, in "Publ. Mat.", 1991, vol. 35, n^o 1, p. 155–168.
- [65] S. JAFFARD. *The multifractal nature of Lévy processes*, in "Probab. Theory Related Fields", 1999, vol. 114, n^o 2, p. 207–227, <http://dx.doi.org/10.1007/s004400050224>.
- [66] H. KEMPKA. *2-Microlocal Besov and Triebel-Lizorkin Spaces of Variable Integrability*, in "Rev. Mat. Complut.", 2009, vol. 22, n^o 1, p. 227–251.
- [67] D. KHOSHNEVISAN. *Multiparameter Processes: an introduction to random fields*, Springer, 2002.
- [68] D. KHOSHNEVISAN. *Multiparameter processes: An Introduction to Random Fields*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2002, xx+584.
- [69] J. LAMPERTI. *Semi-stable stochastic processes*, in "Trans. Am. Math. Soc.", 1962, vol. 104, p. 62-78.
- [70] J. LI, F. NEKKA. *A Pharmacokinetic Formalism Explicitly Integrating the Patient Drug Compliance*, in "J. Pharmacokinet. Pharmacodyn.", 2007, vol. 34, n^o 1, p. 115–139.
- [71] J. LI, F. NEKKA. *A probabilistic approach for the evaluation of pharmacological effect induced by patient irregular drug intake*, in "J. Pharmacokinet. Pharmacodyn.", 2009, vol. 36, n^o 3, p. 221–238.
- [72] M. B. MARCUS, J. ROSEN. *Markov Processes, Gaussian Processes and Local Times*, Cambridge University Press, 2006.
- [73] T. MORI, H. OODAIRA. *The law of the iterated logarithm for self-similar processes represented by multiple Wiener integrals*, in "Probab. Theory Relat. Fields", 1986, vol. 71, n^o 3, p. 367–391, <http://dx.doi.org/10.1007/BF01000212>.
- [74] G. SAMORODNITSKY, M. TAQQU. *Stable Non-Gaussian Random Processes*, Chapman and Hall, 1994.
- [75] S. STOEV, M. TAQQU. *Stochastic properties of the linear multifractional stable motion*, in "Adv. Appl. Probab.", 2004, vol. 36, p. 1085–1115.
- [76] B. VRIJENS, J. URQUHART. *New findings about patient adherence to prescribed drug dosing regimens: an introduction to pharmionics*, in "Eur. J. Hosp. Pharm. Sci.", 2005, vol. 11, n^o 5, p. 103–106.
- [77] B. VRIJENS, J. URQUHART. *Patient adherence to prescribed antimicrobial drug dosing regimens*, in "J. Antimicrob. Chemother.", 2005, vol. 55, p. 616–627.
- [78] Y. XIAO, H. LIN. *Dimension properties of sample paths of self-similar processes*, in "Acta Math. Sinica (N.S.)", 1994, vol. 10, n^o 3, p. 289–300, A Chinese summary appears in Acta Math. Sinica 38 (1995), no. 4, 576.