IN PARTNERSHIP WITH:
CNRS

Ecole nationale supérieure des techniques avancées

Ecole Polytechnique

Activity Report 2013

Project-Team COMMANDS

Control, Optimization, Models, Methods and Applications for Nonlinear Dynamical Systems

IN COLLABORATION WITH: Centre de Mathématiques Appliquées (CMAP), Unité de Mathématiques Appliquées (UMA - ENSTA)

RESEARCH CENTER
Saclay - Île-de-France

THEME
Optimization and control of dynamic systems

Table of contents

Project-Team COMMANDS

Keywords: System Analysis And Control, Nonlinear Control, Optimal Control, Numerical Methods, Nonsmooth Analysis

Creation of the Project-Team: 2009 January 01.

1. Members

Research Scientists
Joseph Frédéric Bonnans [Team leader, Inria, Senior Researcher, HdR]
Pierre Martinon [Inria, Researcher]

Faculty Members
Hasnaa Zidani [Team coleader, ENSTA, Associate Professor, HdR]
Anna Désilles [DGA, until May 2013]

External Collaborators
Nicolas Forcadel [Insa Rouen]
Olivier Bokanowski [Univ. Paris VII]

Engineers
Daphné Giorgi [Inria]
Stéphan Maindrault [Inria, from Apr 2013]

PhD Students
Mohamed Assellaou [Eleve Officier (Maroc) + bourse Ensta]
Xavier Dupuis [ENS Lyon, until Nov. 2013]
Nicolas Grebille [EDF, CIFRE]
Cristopher Hermosilla [Inria, FP7 ITN SADCO project]
Benjamin Heymann [Ecole Polytechnique, from Oct. 2013]
Laurent Pfeiffer [Ecole Polytechnique, until Nov. 2013]
Athena Picarelli [Inria, FP7 ITN SADCO project]
Zhiping Rao [Ecole Polytechnique, FP7 ITN SADCO project, until Dec. 2013]

Post-Doctoral Fellow
Philip Graber [Inria, until May 2013]

Visiting Scientists
Adriano Festa [Post-doc ICL, from Mar 2013 until Aug 2013]
Bean San Goh [Prof. U. Curtin, Miri, Malaysia, from Feb 2013 until Mar 2013]
Ibtissam Medarhri [Inria, from Sep 2013 until Oct 2013]

Administrative Assistants
Wallis Chaussebourg [Inria, until Aug. 2013]
Jessica Gameiro [Inria, since Oct 2013]

Others
Arthur Marly [ENS Lyon, internship, from May 2013 until Jul 2013]
Teresa Scarinci [Inria, internship, FP7 ITN SADCO project, since Oct 2013]

2. Overall Objectives

2.1. Scientific directions

Commands is a team devoted to dynamic optimization, both for deterministic and stochastic systems. This includes the following approaches: trajectory optimization, deterministic and stochastic optimal control, stochastic programming, dynamic programming and Hamilton-Jacobi-Bellman equation.

Our aim is to derive new and powerful algorithms for solving numerically these problems, with applications in several industrial fields. While the numerical aspects are the core of our approach it happens that the study of convergence of these algorithms and the verification of their well-posedness and accuracy raises interesting and difficult theoretical questions, such as, for trajectory optimization: qualification conditions and second-order optimality condition, well-posedness of the shooting algorithm, estimates for discretization errors; for the Hamilton-Jacobi-Bellman approach: accuracy estimates, strong uniqueness principles when state constraints are present, for stochastic programming problems: sensitivity analysis.

2.2. Industrial impact

For many years the team members have been deeply involved in various industrial applications, often in the framework of PhD theses or of postdocs. The Commands team itself has dealt since its foundation in 2007 with several types of applications:

- Space vehicle trajectories, in collaboration with CNES, the French space agency.
- Aeronautics, in collaboration with the startup Safety Line.
- Production, management, storage and trading of energy resources (in collaboration with EDF, GDF and TOTAL).
- Energy management for hybrid vehicles (in collaboration with Renault).

We give more details in the Bilateral contracts section.

2.3. Highlights of the Year

In collaboration with L. Giraldi and M. Zopello, we started in 2013 to study the optimal swimming strategies for micro-swimmers. Our approach allows us to solve the optimal control problem without making restrictive assumptions on the shape of the swimming movements. The first numerical results on the 3-link swimmer indicate the existence of a periodic stroke with a better displacement speed than the canonical stroke presented by Purcell in 1977. Further directions include optimal design of micro-swimmers and comparing our simulations to the movement of live micro-organisms.

In collaboration with CNES, a trajectory optimization problem for Ariane 5 was studied and analyzed by HJB approach. In this study, the flight model is considered in dimension 6 without simplification. The problem consists in maximizing the payload to steer the launcher from the launch base (Kourou) to the GEO orbit. The mission includes ballistic phases and the optimization also encompasses the intermediate GTO orbit parameters. The optimization criterion is the mass of the payload to be injected on the GEO.

Finally, the team completed 3 PhD and 4 patents in 2013.

3. Research Program

3.1. Historical aspects

The roots of deterministic optimal control are the "classical" theory of the calculus of variations, illustrated by the work of Newton, Bernoulli, Euler, and Lagrange (whose famous multipliers were introduced in [78]), with improvements due to the "Chicago school", Bliss [51] during the first part of the 20th century, and by the notion of relaxed problem and generalized solution (Young [86]).

Trajectory optimization really started with the spectacular achievement done by Pontryagin's group [84] during the fifties, by stating, for general optimal control problems, nonlocal optimality conditions generalizing those of Weierstrass. This motivated the application to many industrial problems (see the classical books by Bryson and Ho [58], Leitmann [80], Lee and Markus [79], Ioffe and Tihomirov [73]). Since then, various theoretical achievements have been obtained by extending the results to nonsmooth problems, see Aubin [47], Clarke [59], Ekeland [66].

Dynamic programming was introduced and systematically studied by R. Bellman during the fifties. The HJB equation, whose solution is the value function of the (parameterized) optimal control problem, is a variant of the classical Hamilton-Jacobi equation of mechanics for the case of dynamics parameterized by a control variable. It may be viewed as a differential form of the dynamic programming principle. This nonlinear first-order PDE appears to be well-posed in the framework of *viscosity solutions* introduced by Crandall and Lions [61], [62], [60]. These tools also allow to perform the numerical analysis of discretization schemes. The theoretical contributions in this direction did not cease growing, see the books by Barles [49] and Bardi and Capuzzo-Dolcetta [48].

3.2. Trajectory optimization

The so-called *direct methods* consist in an optimization of the trajectory, after having discretized time, by a nonlinear programming solver that possibly takes into account the dynamic structure. So the two main problems are the choice of the discretization and the nonlinear programming algorithm. A third problem is the possibility of refinement of the discretization once after solving on a coarser grid.

In the *full discretization approach*, general Runge-Kutta schemes with different values of control for each inner step are used. This allows to obtain and control high orders of precision, see Hager [70], Bonnans [54]. In an interior-point algorithm context, controls can be eliminated and the resulting system of equation is easily solved due to its band structure. Discretization errors due to constraints are discussed in Dontchev et al. [65]. See also Malanowski et al. [81].

In the *indirect* approach, the control is eliminated thanks to Pontryagin's maximum principle. One has then to solve the two-points boundary value problem (with differential variables state and costate) by a single or multiple shooting method. The questions are here the choice of a discretization scheme for the integration of the boundary value problem, of a (possibly globalized) Newton type algorithm for solving the resulting finite dimensional problem in IR^n (n is the number of state variables), and a methodology for finding an initial point.

For state constrained problems or singular arcs, the formulation of the shooting function may be quite elaborate [52], [53], [46]. As initiated in [69], we focus more specifically on the handling of discontinuities, with ongoing work on the geometric integration aspects (Hamiltonian conservation).

3.3. Hamilton-Jacobi-Bellman approach

This approach consists in calculating the value function associated with the optimal control problem, and then synthesizing the feedback control and the optimal trajectory using Pontryagin's principle. The method has the great particular advantage of reaching directly the global optimum, which can be very interesting, when the problem is not convex.

Characterization of the value function From the dynamic programming principle, we derive a characterization of the value function as being a solution (in viscosity sense) of an Hamilton-Jacobi-Bellman equation, which is a nonlinear PDE of dimension equal to the number n of state variables. Since the pioneer works of Crandall and Lions [61], [62], [60], many theoretical contributions were carried out, allowing an understanding of the properties of the value function as well as of the set of admissible trajectories. However, there remains an important effort to provide for the development of effective and adapted numerical tools, mainly because of numerical complexity (complexity is exponential with respect to n).

Numerical approximation for continuous value function Several numerical schemes have been already studied to treat the case when the solution of the HJB equation (the value function) is continuous. Let us quote for example the Semi-Lagrangian methods [68], [67] studied by the team of M. Falcone (La Sapienza, Rome), the high order schemes WENO, ENO, Discrete galerkin introduced by S. Osher, C.-W. Shu, E. Harten [71], [72], [72], [82], and also the schemes on nonregular grids by R. Abgrall [45], [44]. All these schemes rely on finite differences or/and interpolation techniques which lead to numerical diffusions. Hence, the numerical solution is unsatisfying for long time approximations even in the continuous case.

One of the (nonmonotone) schemes for solving the HJB equation is based on the Ultrabee algorithm proposed, in the case of advection equation with constant velocity, by Roe [85] and recently revisited by Després-Lagoutière [64], [63]. The numerical results on several academic problems show the relevance of the antidiffusive schemes. However, the theoretical study of the convergence is a difficult question and is only partially done.

Optimal stochastic control problems occur when the dynamical system is uncertain. A decision typically has to be taken at each time, while realizations of future events are unknown (but some information is given on their distribution of probabilities). In particular, problems of economic nature deal with large uncertainties (on prices, production and demand). Specific examples are the portfolio selection problems in a market with risky and non-risky assets, super-replication with uncertain volatility, management of power resources (dams, gas). Air traffic control is another example of such problems.

Nonsmoothness of the value function. Sometimes the value function is smooth (e.g. in the case of Merton's portfolio problem, Oksendal [87]) and the associated HJB equation can be solved explicitly. Still, the value function is not smooth enough to satisfy the HJB equation in the classical sense. As for the deterministic case, the notion of viscosity solution provides a convenient framework for dealing with the lack of smoothness, see Pham [83], that happens also to be well adapted to the study of discretization errors for numerical discretization schemes [76], [50].

Numerical approximation for optimal stochastic control problems. The numerical discretization of second order HJB equations was the subject of several contributions. The book of Kushner-Dupuis [77] gives a complete synthesis on the Markov chain schemes (i.e Finite Differences, semi-Lagrangian, Finite Elements, ...). Here a main difficulty of these equations comes from the fact that the second order operator (i.e. the diffusion term) is not uniformly elliptic and can be degenerated. Moreover, the diffusion term (covariance matrix) may change direction at any space point and at any time (this matrix is associated the dynamics volatility).

For solving stochastic control problems, we studied the so-called Generalized Finite Differences (GFD), that allow to choose at any node, the stencil approximating the diffusion matrix up to a certain threshold [57]. Determining the stencil and the associated coefficients boils down to a quadratic program to be solved at each point of the grid, and for each control. This is definitely expensive, with the exception of special structures where the coefficients can be computed at low cost. For two dimensional systems, we designed a (very) fast algorithm for computing the coefficients of the GFD scheme, based on the Stern-Brocot tree [56].

4. Software and Platforms

4.1. Bocop

Participants: Pierre Martinon [corresponding author], Daphné Giorgi, Joseph Frédéric Bonnans.

Web page: http://bocop.org

The Bocop project aims to develop an open-source toolbox for solving optimal control problems, with collaborations with industrial and academic partners. Optimal control (optimization of dynamical systems governed by differential equations) has numerous applications in transportation, energy, process optimization, and biology. The software reuses some packages from the COIN-OR library, in particular the well-known nonlinear programming solver Ipopt, features a user-friendly interface and can be deployed on Windows / Mac / Linux.

The project is supported by Inria with the recruitment of Vincent Grelard as developer in 2010-2012, and then Daphné Giorgi since October 2012. The first prototype was released at the end of 2011, Bocop is currently at version 1.1.4 and has been downloaded more than 700 times. The software was first succesfully tested on several academic problems, see [55] available on http://bocop.org. Starting in 2012, several research collaborations were initiated in fields such as bio-reactors for energy production ([30], [27]), swimming micro-robots ([39]), and quantum control for medical imaging ([25]). Bocop was also featured during our participation in the Imatch "Optimisation and Control" in october, which resulted in a contract with the startup Safety Line (aeronautics).

Bocop auto-assessment according to Inria notice: A3up4, SO3, SM3, EM3up4, SDL4up5

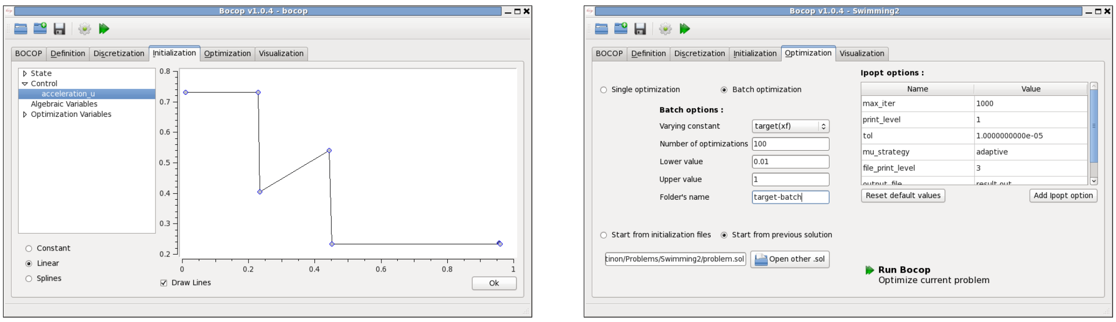

Figure 1. BOCOP

4.2. CollAv

Participants: Hasnaa Zidani [corresponding author], Olivier Bokanowski, Anna Désilles.

This software simulates the evolution of controlled dynamical systems (possibly under uncertainties). The numerical algorithm here is based on HJB or viability approaches, and allows the design of optimal planning strategies (according to a criterion determined by the user: time, energy, ...). It also provides conflict resolution and avoidance of collisions with fixed or moving obstacles. So far, the software is used in collaboration with DGA for avoidance collision of UaVs, and by Volkswagen in some studies related to collision avoidance of cars.

4.3. OCOPHyS

Participant: Hasnaa Zidani [corresponding author].

This is a software for optimisation-based controller design for operating in different regimes or modes of operation. The software can be used, for example, to determine the optimal management for hybrid vehicles or hybrid engines with multiple energy sources. However, the methods used in software are still quite general and can be used in many applications.

4.4. BiNoPe-HJ

Participants: Hasnaa Zidani [corresponding author], Olivier Bokanowski, Anna Désilles.

Web page: http://www.ensta-paristech.fr/ zidani/BiNoPe-HJ

This project aims at developping sequential and parallel MPI/openMP C++ solvers for the approximation of Hamilton-Jacobi-Bellman (HJB) equations in a d-dimensional space. The main goal is to provide an HJB solvers that can work in dimension d (limited by the machine's capacity). The solver outputs can be visualized with Matlab or Paraview (via VTK files).

The development of the HJB Solver has been initiated under a partnership between COMMANDS and the SME HPC-project in the period between December 2009 to November 2011. Currently, it is still maintained and improved by COMMANDS.

In 2012, two versions were released:

- HJB-SEQUENTIAL-REF: sequential version that can run on any machine
- HJB-PARALLEL-REF: parallel version that can run only on multi-core architectures.

4.5. Shoot

Participant: Pierre Martinon [corresponding author].

Shoot was designed for the resolution of optimal control problems via indirect methods (necessary conditions, Pontryagin's Maximum Principle). Such methods transform the original problem into finding a zero of a certain shooting function. The package offers several choices of integrators and solvers, and can handle control discontinuities. Features also include the use of variational equations to compute the Jacobian of the shooting function, as well as homotopy and grid shooting techniques for easier initialization. Shoot is an academic software, and was used during several research contracts with the CNES (french space agency).

5. New Results

5.1. Optimality conditions in Pontryagin form for optimal control problems

Participants: Joseph Frédéric Bonnans, Xavier Dupuis, Laurent Pfeiffer.

5.1.1. Necessary conditions

In the paper [31], we state and prove first- and second-order necessary conditions in Pontryagin form for optimal control problems with pure state and mixed control-state constraints. We say that a Lagrange multiplier of an optimal control problem is a Pontryagin multiplier if it is such that Pontryagin's minimum principle holds, and we call optimality conditions in Pontryagin form those which only involve Pontryagin multipliers. Our conditions rely on a technique of partial relaxation, and apply to Pontryagin local minima.

5.1.2. Sufficient conditions

In the paper [32], we consider sufficient conditions. More precisely, given a reference feasible trajectory of an optimal control problem, we say that the quadratic growth property for bounded strong solutions holds if the cost function of the problem has a quadratic growth over the set of feasible trajectories with a bounded control and with a state variable sufficiently close to the reference state variable. Our sufficient second-order optimality conditions in Pontryagin form ensure this property and ensure *a fortiori* that the reference trajectory is a bounded strong solution. Our proof relies on a decomposition principle, which is a particular second-order expansion of the Lagrangian of the problem.

5.1.3. Shooting Approach to Optimal Control Problems

Participant: Joseph Frédéric Bonnans.

In the paper [24] we give an overview of the shooting technique for solving deterministic optimal control problems. This approach allows to reduce locally these problems to a finite dimensional equation. We first recall the basic idea, in the case of unconstrained or control constrained problems, and show the link with second-order optimality conditions and the analysis or discretization errors. Then we focus on two cases that are now better understood: state constrained problems, and affine control systems. We end by discussing extensions to the optimal control of a parabolic equation.

5.2. Applications of deterministic optimal control problems

5.2.1. Optimization of running strategies based on anaerobic energy and variations of velocity

Participant: Joseph Frédéric Bonnans.

In the report [29] we present new models, numerical simulations and rigorous analysis for the optimization of the velocity in a race. In a seminal paper, Keller [74], [75] explained how a runner should determine his speed in order to run a given distance in the shortest time. We extend this analysis, based on the equation of motion and aerobic energy, to include a balance of anaerobic energy (or accumulated oxygen deficit) and an energy recreation term when the speed decreases. We also take into account that when the anaerobic energy gets too low, the oxygen uptake cannot be maintained to its maximal value. Using optimal control theory, we obtain a proof of Keller's optimal race, and relate the problem to a relaxed formulation, where the propulsive force represents a probability distribution rather than a value function of time. Our analysis leads us to introduce a bound on the variations of the propulsive force to obtain a more realistic model which displays oscillations of the velocity. Our numerical simulations qualitatively reproduce quite well physiological measurements on real runners. We show how, by optimizing over a period, we recover these oscillations of speed. We point out that our numerical simulations provide in particular the exact instantaneous anaerobic energy used in the exercise.

5.2.2. Optimal control of leukemic cell population dynamics

Participant: Xavier Dupuis.

In the paper [33] we discuss the optimal co-administration of two drugs for some acute myeloid leukemias (AML), and we are looking for in vitro protocols as a first step. This issue can be formulated as an optimal control problem. The dynamics of leukemic cell populations in culture is given by age-structured partial differential equations, which can be reduced to a system of delay differential equations, and where the controls represent the action of the drugs. The objective function relies on eigenelements of the uncontrolled model and on general relative entropy, with the idea to maximize the efficiency of the protocols. The constraints take into account the toxicity of the drugs. We present in this paper the modeling aspects, as well as theoretical and numerical results on the optimal control problem that we get.

5.2.3. Contrast imaging problem in nuclear magnetic resonance

Participant: Pierre Martinon.

In collaboration with team McTAO (Sophia), we studied in [25] and [36] the contrast imaging problem in nuclear magnetic resonance, modeled as Mayer problem in optimal control. The optimal solution can be found as an extremal, solution of the Maximum Principle and analyzed with the techniques of geometric control. A first synthesis of locally optimal solutions is given in the single-input case, with some preliminary results in the bi-input case. We conducted a comprehensive numerical investigation of the problem, using a combination of indirect shooting (HAMPATH software) and direct method (BOCOP), with a moment-based (LMI) technique to estimate the global optimum.

5.2.4. Optimizing the anaerobic digestion of microalgae in a coupled process

Participant: Pierre Martinon.

In collaboration with the Inra-Inria team MODEMIC (Montpellier), we studied in [30] a bio-reactor system describing the coupling of a culture of micro-algae and an anaerobic digester. Our aim is to optimize the production of methane in the digester during a certain number of days with respect to the dilution rate (the input flow of micro-algae in the digester). The mathematical model for the dynamics of the two reactors takes into account a periodic day-night model of the light in the culture of micro-algae, and a chemostat model for the digester. We first prove existence and attraction of periodic solutions for a one day period, and we apply Pontryagin's Maximum Principle (PMP) in order to characterize optimal controls. We provide numerical simulations for different light models, by a direct method that we refine using an indirect shooting. We also investigate the dependence of the optimal cost with respect to the ratio of the volumes of the two tanks. Finally, we investigate the optimal strategies over a large number of days without periodic constraints, and compared the mean cost to the optimal cost over one period.

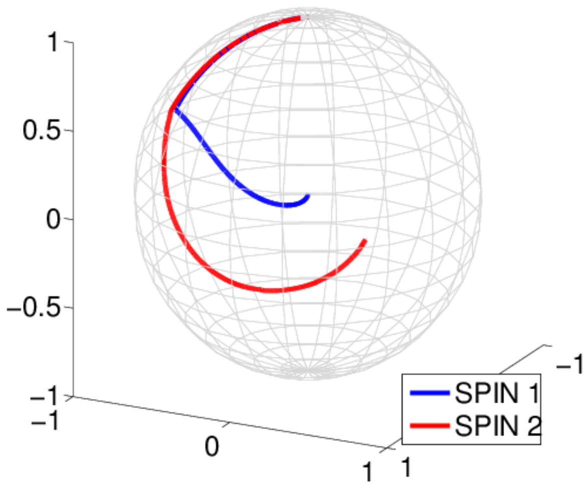

Figure 2. Contrast in quantum control for NMR - Oxygenated / deoxygenated blood

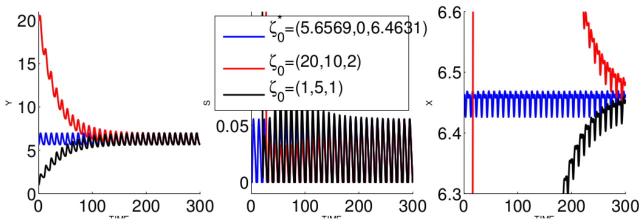

Figure 3. Coupled bio-reactor for micro-algae digestion - Attraction property

5.2.5. Design of optimal experiments for parameter estimation of microalgae growth models

Participant: Pierre Martinon.

In collaboration with team BIOCORE (Sophia), we investigated in [27] techniques of Optimal Experiment Design for microalgae growth models. In order to have microalgae growth models that are useful for prediction and process optimization, reliable parameters need to be provided. This reliability implies a careful design of experiments that can be exploited for parameter estimation. OED techniques can provide guidelines for the design of experiments with high informative content that allow an accurate parameter estimation. We study a real experimental device devoted to evaluate the effect of temperature and light on microalgae growth. On the basis of a mathematical model of the experimental system, the optimal experiment design problem was solved as an optimal control problem. E-optimal experiments were obtained by using two discretization approaches, namely sequential and simultaneous. The results showed that an adequate parameterization of the experimental inputs provided optimal solutions very close to those provided by the simultaneous discretization. Simulation results showed the relevance of determining optimal experimental inputs for achieving an accurate parameter estimation.

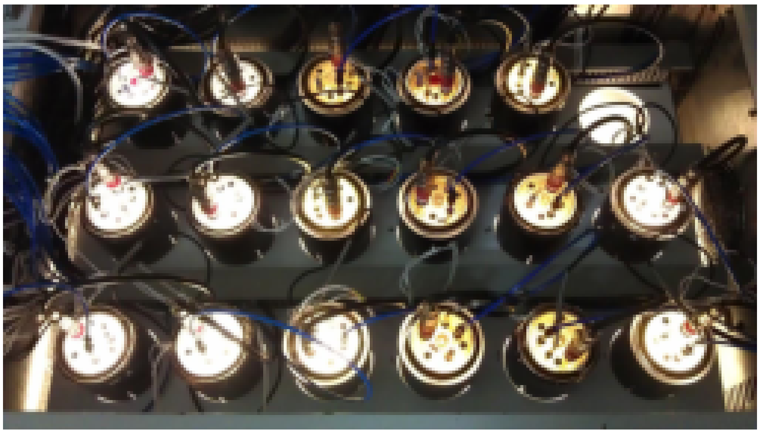

Figure 4. Experimental apparatus for the study of micro-algae growth (Ifremer)

5.2.6. Controllability and optimal strokes for N-link microswimmer

Participant: Pierre Martinon.

In [39] we focus on the N-link swimmer, a generalization of the classical Purcell swimmer. We use the simplification of the Resistive Force Theory to derive the motion equation for the swimmer in a fluid with a low Reynolds number. We prove that the swimmer is controllable in the whole plane when it is composed by more than 3 sticks and for almost every set of stick lengths. As a direct result, we show that there exists an optimal swimming strategy which leads to minimize the time to reach a desired configuration. Numerical experiments on the case of the Purcell swimmer suggest that the optimal strategy is periodic, i.e. composed of a sequence of identical strokes. Our results indicate that this candidate for an optimal stroke indeed gives a better speed than the classical Purcell stroke. Future directions for this work include the design of robotic micro-swimmers, as well as investigation of the movement of swimming micro-organisms.

5.3. Hamilton-Jacobi (HJ) approach

5.3.1. Dynamic programming and error estimates for stochastic control with Max cost

Participants: Olivier Bokanowski, Athena Picarelli, Hasnaa Zidani.

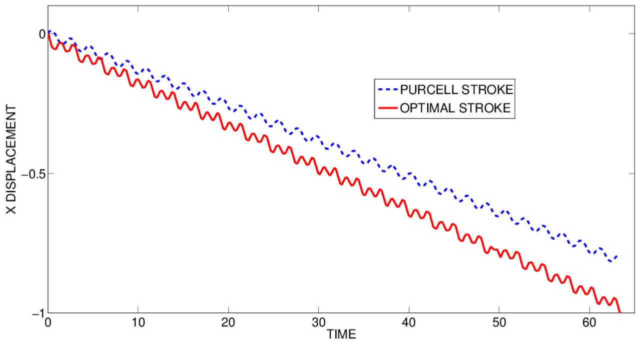

Figure 5. Purcell (3-link) swimmer - Purcell vs optimal stroke

The paper [35] is concerned with stochastic optimal control for a running maximum cost. A direct approach based on dynamic programming techniques is studied leading to the characterization of the value function as the unique viscosity solution of a second order Hamilton-Jacobi-Bellman (HJB) equation with an oblique derivative boundary condition. A general numerical scheme is proposed and a convergence result is provided. Error estimates are obtained for the semi-Lagrangian scheme. These results can apply to the case of lookback options in finance. Moreover, optimal control problems with maximum cost arise in the characterization of the reachable sets for a system of controlled stochastic differential equations. Some numerical simulations on examples of reachable analysis are included to illustrate our approach.

5.3.2. Optimal feedback control of undamped wave equations by solving a HJB equation

Participant: Hasnaa Zidani.

An optimal finite-time horizon feedback control problem for (semi linear) wave equations is studied in [42]. The feedback law can be derived from the dynamic programming principle and requires to solve the evolutionary Hamilton-Jacobi-Bellman (HJB) equation. Classical discretization methods based on finite elements lead to approximated problems governed by ODEs in high dimensional space which makes infeasible the numerical resolution by HJB approach. In the present paper, an approximation based on spectral elements is used to discretize the wave equation. The effect of noise is considered and numerical simulations are presented to show the relevance of the approach

5.3.3. Transmission conditions on interfaces for Hamilton-Jacobi-Bellman equations

Participants: Hasnaa Zidani, Zhiping Rao.

The works [43], [28] deal with deterministic control problems where the dynamic and the running cost can be completely different in two (or more) complementary domains of the space IR^N. As a consequence, the dynamics and running cost present discontinuities at the interfaces of these domains. This leads to a complex interplay that has to be analyzed among transmission conditions to "glue" the propagation of the value function on the interfaces. Several questions arise: how to define properly the value function(s) and what is (are) the right Bellman Equation(s) associated with this problem?. In the case of a simple geometry (namely when the space IR^N is partitioned into two subdomains separated with an interface which is assumed to be a regular hypersurface without any connectedness requirement), [43] discuss different conditions on the hyperplane where the dynamic and the running cost are discontinuous, and the uniqueness properties of the Bellman problem are studied. In this paper it is used a dynamical approach, namely instead of working with test functions, the accent is put on invariance properties of an augmented dynamics related to the integrated control system. The comparison principle is accordingly based, rather than on (semi)continuity of the Hamiltonian appearing in the Hamilton–Jacobi–Bellman equation, on some weak separation properties of this dynamics with respect to the stratification. A more general situation where the space is partionned on several domains is also analyzed in [28].

5.3.4. *Singular perturbation of optimal control problems on multi-domains*

Participants: Nicolas Forcadel, Hasnaa Zidani.

The goal of the paper[38] is to study a singular perturbation problem in the framework of optimal control on multi-domains. We consider an optimal control problem in which the controlled system contains a fast and a slow variables. This problem is reformulated as an Hamilton-Jacobi-Bellman (HJB) equation. The main difficulty comes from the fact that the fast variable lives in a multi-domain. The geometric singularity of the multi-domains leads to the discontinuity of the Hamiltonian. Under a controllability assumption on the fast variables, the limit equation (as the velocity of the fast variable goes to infinity) is obtained via a PDE approach and by means of the tools of the control theory.

5.3.5. *Optimal control of first order HJ equations with linearly bounded Hamiltonian*

Participant: Philip Graber.

In [40], we consider the optimal control of solutions of first order Hamilton-Jacobi equations, where the Hamiltonian is convex with linear growth. This models the problem of steering the propagation of a front by constructing an obstacle. We prove existence of minimizers to this optimization problem as in a relaxed setting and characterize the minimizers as weak solutions to a mean field game type system of coupled partial differential equations. Furthermore, we prove existence and partial uniqueness of weak solutions to the PDE system. An interpretation in terms of mean field games is also discussed.

5.3.6. *Zubov's equation for state-constrained perturbed nonlinear systems*

Participant: Hasnaa Zidani.

The paper [41] gives a characterization of the uniform robust domain of attraction for a nite non-linear controlled system subject to perturbations and state constraints. We extend the Zubov approach to characterize this domain by means of the value function of a suitable in nite horizon state-constrained control problem which at the same time is a Lyapunov function for the system. We provide associated Hamilton-Jacobi-Bellman equations and prove existence and uniqueness of the solutions of these generalized Zubov equations.

5.3.7. *Numerical methods for chance-constrained stochastic optimal control problems*

Participant: Laurent Pfeiffer.

In Laurent Pfeiffer's PhD, we study stochastic optimal control problems with a probability constraint on the final state. This constraint must be satisfied with a probability greater or equal than a given level. We analyse and compare two approaches for discrete-time problems: a first one based on a dynamic programming principle and a second one using Lagrange relaxation. These approaches can be used for continuous-time problems, for which we give numerical illustrations.

6. Bilateral Contracts and Grants with Industry

6.1. Safety Line

Following the "iMatch Contrôle Optimisation" event held at Inria Saclay on October 23rd (2012), a collaboration was initiated between COMMANDS and the startup Safety Line (http ://www.safety-line.fr), with a first contract on optimizing the ascent phase for commercial planes. A crucial aspect of this work is the identification of accurate and reliable models for the aerodynamic and thrust forces acting on the plane. For this study our partners at Safety Line provide us access to data recorded during several thousands of actual commercial flights, and COMMANDS recruited Stephan Maindrault as engineer to work on this project.

6.2. CNES

This contract between CNES and ENSTA lasted from February to December 2013, and was devoted to trajectory global optimization for an Ariane 5 launcher, using HJB techniques. The optimization was on the whole launch, including ballistic phases and the parameters of the intermediate GTO orbit, while maximizing the payload mass.

7. Partnerships and Cooperations

7.1. National Initiatives

7.1.1. DGA

Participants: Olivier Bokanowski, Anna Désilles, Hasnaa Zidani.

This project is a collaboration in the framework of a 3-year (2012-2015) research program funded by DGA. The title of the project is "Problèmes de commande optimale pour des systèmes non-linéaires en présence d'incertitudes et sous contraintes de probabilité de succès".

7.1.2. ANR HJNet

Participants: Olivier Bokanowski, Zhiping Rao, Hasnaa Zidani.

The team is part of the collaborative project HJNet funded by the French National Research Agency (ANR-12-BS01-0008-01). It started in January 2013 and will end in December 2013. Website: http://hjnet.math.cnrs.fr

7.2. European Initiatives

7.2.1. FP7 Projects

7.2.1.1. SADCO

Instrument: Initial Training Network

Duration: January 2011 - December 2014

Coordinator: Inria

Partner: Univ. of Louvain, Univ. Bayreuth, Univ. Porto, Univ. Rome - La Sapienza, ICL, Astrium-Eads, Astos solutions, Volkswagen, Univ. Padova, Univ. Pierre et Marie Curie.

Inria contact: Hasnaa Zidani

Abstract: Optimisation-based control systems concern the determination of control strategies for complex, dynamic systems, to optimise some measures of best performance. It has the potential for application to a wide range of fields, including aerospace, chemical processing, power systems control, transportation systems and resource economics. It is of special relevance today, because optimization provides a natural framework for determining control strategies, which are energy efficient and respect environmental constraints. The multi-partner initial training network SADCO aims at: Training young researchers and future scientific leaders in the field of control theory with emphasis on two major themes sensitivity of optimal strategies to changes in the optimal control problem specification, and deterministic controller design; Advancing the theory and developing new numerical methods; Conveying fundamental scientific contributions within European industrial sectors.

See: http://itn-sadco.inria.fr

7.3. International Initiatives

7.3.1. Inria Associate Teams

7.3.1.1. OCONET

Title: Optimization and control in network economics

Inria principal investigator: Frédéric Bonnans

International Partner (Institution - Laboratory - Researcher):

University of Chile (Chile) - Center for Mathematical Modeling - Joseph Frédéric Bonnans

Duration: 2012 - 2014

See also: http://www.cmm.uchile.cl/EA_OCONET

Limited resources in telecommunication, energy, gas and water supply networks, lead to multi-agent interactions that can be seen as games or economic equilibrium involving stochastic optimization and optimal control problems. Interaction occurs within a network, where decisions on what to produce, consume, trade or plan, are subject to constraints imposed by node and link capacities, risk, and uncertainty, e.g. the capacity of generators and transmission lines; capacity of pipeline in gas supply; switches and antennas in telecommunication. At the same time, nonlinear phenomena arise from price formation as a consequence of demand-supply equilibria or multi-unit auction processes in the case of energy and telecommunication. We will focus first in this project in electricity markets in which there are producers/consumers PCs, and an agent called ISO (Independent system operator) in charge of the management of the network. One major application we have in mind is the one of smart (electrical) grids, in view of the increased use of renewable energies, that is, a massive entry of wind, geothermal, solar in particular.

7.3.2. Inria International Labs

- The team is involved in the "Energy Optimization" group of the Inria research center in Chile (CIRIC). Several visits to Chile were conducted in relation with this project.

7.4. International Research Visitors

7.4.1. Visits of International Scientists

- Prof. B.S. Goh, Curtin University, Miri, Malaysia; two weeks in February.
- M.S. Aronna, Rosario University, Argentina; one month (February and November).

7.4.1.1. Internships

- Arthur Marly, second year student of ENS Lyon. Two months intership. Subject: optimal control of populations with state constraints. Supervisor: F. Bonnans.

8. Dissemination

8.1. Scientific Animation

- F. Bonnans is Corresponding Editor of “ESAIM:COCV” (Control, Optimization and Calculus of Variations), and Associate Editor of “Applied Mathematics and Optimization”, “Optimization, Methods and Software”, and “Series on Mathematics and its Applications, Annals of The Academy of Romanian Scientists”.
- F. Bonnans is chairman of the SMAI-MODE group (the optimization group of the French Applied Mathematics Society) until June 2013.

8.2. Teaching - Supervision - Juries

- F. Bonnans: Optimal control, 7h, M2, Ensta, France.
- F. Bonnans: Continuous Optimization, 18h, M2, Ecole Polytechnique and U. Paris 6, France.

- F. Bonnans: Numerical analysis of partial differential equations arising in finance and stochastic control, 24h, M2, Ecole Polytechnique and U. Paris 6, France.
- H. Zidani: Optimal control, 14h, M2, Ensta, France.
- H. Zidani: Numerical methods for front propagation, 21h, M2, Ensta France

  PhD: Xavier Dupuis, Optimal control with or witout memories. ended Nov. 2013, F. Bonnans.

  PhD : Laurent Pfeiffer, Sensitivity analysis for optimal control problems; Stochastic optimal control with probability constraints. ended Nov. 2013, F. Bonnans.

  PhD: Zhiping Rao, Hamilton-Jacobi equations with discontinuous coefficients. Ended Dec. 2013, H. Zidani and N. Forcadel.

  PhD in progress : Imène Ben-Latifa, Optimal multiple stopping and valuation of swing options in jump models. Oct. 2010, F. Bonnans and M. Mnif (ENIT, Tunis).

  PhD in progress: Athena Picarelli, First and Second Order Hamilton-Jacobi equations for State-Constrained Control Problems. Nov. 2011, O. Bokanowski and H. Zidani

  PhD in progress: Cristopher Hermosilla, Feedback controls and optimal trajectories. Nov. 2011, H. Zidani.

  PhD in progress: Mohamed Assellaou, Reachability analysis for stochastic controlled systems. Oct. 2011, O. Bokanowski and H.Zidani.

  PhD in progress: Benjamin Heymann, Dynamic optimization with uncertainty; application to energy production. Oct. 2013, F. Bonnans.

8.3. Popularization

Pierre Martinon presented an “Unithé ou Café” talk about optimal control on November 8th, entitled “Quand le mieux n’est pas l’ennemi du bien”.

9. Bibliography

Major publications by the team in recent years

[1] O. BOKANOWSKI, B. BRUDER, S. MAROSO, H. ZIDANI. *Numerical approximation for a superreplication problem under gamma constraints*, in "SIAM. Num. Analysis.", 2009, vol. 47(3), pp. 2289–2320

[2] O. BOKANOWSKI, N. MEGDICH, H. ZIDANI. *Convergence of a non-monotone scheme for Hamilton-Jacobi-Bellman equations with discontinuous data*, in "Numerische Mathematik / Numerical Mathematics", 2010, vol. 115, n° 1, pp. 1–44, http://hal.inria.fr/inria-00193157

[3] J. F. BONNANS, J. C. GILBERT, C. LEMARÉCHAL, C. SAGASTIZÁBAL. , *Numerical Optimization: theoretical and numerical aspects*, Universitext, Springer-Verlag, Berlin, 2006, second edition

[4] J. F. BONNANS, S. MAROSO, H. ZIDANI. *Error estimates for a stochastic impulse control problem*, in "Appl. Math. and Optim.", 2007, vol. 55, n° 3, pp. 327–357

[5] J. F. BONNANS, A. SHAPIRO. , *Perturbation analysis of optimization problems*, Springer-Verlag, New York, 2000

[6] J. F. BONNANS, H. ZIDANI. *Consistency of generalized finite difference schemes for the stochastic HJB equation*, in "SIAM J. Numerical Analysis", 2003, vol. 41, pp. 1008-1021

[7] N. BÉREND, J. F. BONNANS, J. LAURENT-VARIN, M. HADDOU, C. TALBOT. *An Interior-Point Approach to Trajectory Optimization*, in "J. Guidance, Control and Dynamics", 2007, vol. 30, n^o 5, pp. 1228-1238

[8] J. GERGAUD, P. MARTINON. *Using switching detection and variational equations for the shooting method*, in "Optimal Control Applications and Methods", 2007, vol. 28, n^o 2, pp. 95–116

[9] P. MARTINON, J. F. BONNANS, J. LAURENT-VARIN, E. TRÉLAT. *Numerical study of optimal trajectories with singular arcs for an Ariane 5 launcher*, in "J. Guidance, Control, and Dynamics", 2009, vol. 32, n^o 1, pp. 51-55

Publications of the year

Doctoral Dissertations and Habilitation Theses

[10] X. DUPUIS. , *Contrôle optimal d'équations différentielles avec - ou sans - mémoire*, Ecole Polytechnique X, November 2013, http://hal.inria.fr/tel-00914246

[11] L. PFEIFFER. , *Analyse de sensibilité pour des problèmes de commande optimale. Commande optimale stochastique sous contrainte en probabilité*, Ecole Polytechnique X, November 2013, http://hal.inria.fr/pastel-00881119

[12] Z. RAO. , *L'approche Hamilton-Jacobi-Bellman pour des problèmes de contrôle optimal avec des coefficients discontinus*, Ecole Polytechnique X, December 2013, http://hal.inria.fr/pastel-00927358

Articles in International Peer-Reviewed Journals

[13] A. ALTAROVICI, O. BOKANOWSKI, H. ZIDANI. *A general Hamilton-Jacobi framework for nonlinear state-constrained control problems*, in "ESAIM: Control, Optimisation and Calculus of Variations", 2013, vol. 19, n^o 2, pp. 337–357 [*DOI :* 10.1051/COCV/2012011], http://hal.inria.fr/hal-00653337

[14] M. S. ARONNA, J. F. BONNANS, P. MARTINON. *A Shooting Algorithm for Optimal Control Problems with Singular Arcs*, in "Journal of Optimization Theory and Applications", August 2013, vol. 158, n^o 2, pp. 419-459 [*DOI :* 10.1007/S10957-012-0254-8], http://hal.inria.fr/inria-00631332

[15] O. BOKANOWSKI, Y. CHENG, C.-W. SHU. *A discontinuous Galerkin scheme for front propagation with obstacles*, in "Numerische Mathematik", June 2013 [*DOI :* 10.1007/S00211-013-0555-3], http://hal.inria.fr/hal-00653532

[16] J. F. BONNANS. *Optimal control of a semilinear parabolic equation with singular arcs*, in "Optimization, Methods and Software", August 2013, 14 p. [*DOI :* 10.1080/10556788.2013.830220], http://hal.inria.fr/hal-00740698

[17] J. F. BONNANS, C. DE LA VEGA, X. DUPUIS. *First and second order optimality conditions for optimal control problems of state constrained integral equations*, in "Journal of Optimization Theory and Applications", 2013, vol. 159, n^o 1, pp. 1-40 [*DOI :* 10.1007/S10957-013-0299-3], http://hal.inria.fr/hal-00697504

[18] J. F. BONNANS, L. PFEIFFER, O. S. SEREA. *Sensitivity analysis for relaxed optimal control problems with final-state constraints*, in "Nonlinear Analysis: Theory, Methods & Applications", September 2013, vol. 89, pp. 55-80 [*DOI :* 10.1016/J.NA.2013.04.013], http://hal.inria.fr/hal-00702246

[19] J. F. BONNANS, X. TAN. *A model-free no-arbitrage price bound for variance options*, in "Applied Mathematics and Optimization", July 2013, vol. 68, n^{o} 1, pp. 43-73 [*DOI :* 10.1007/s00245-013-9197-1], http://hal.inria.fr/inria-00634387

[20] N. FORCADEL, Z. RAO, H. ZIDANI. *State-constrained Optimal Control Problems of Impulsive Differential Equations*, in "Applied Mathematics and Optimization", 2013 [*DOI :* 10.1007/s00245-013-9193-5], http://hal.inria.fr/hal-00653671

[21] G. GRANATO, H. ZIDANI. *Level-set approach for Reachability Analysis of Hybrid Systems under Lag Constraints*, in "SIAM Journal on Control and Optimization", 2014, http://hal.inria.fr/hal-00735724

[22] C. IMBERT, R. MONNEAU, H. ZIDANI. *A Hamilton-Jacobi approach to junction problems and application to traffic flows*, in "ESAIM - Control Optimisation and Calculus of Variations", 2013, vol. 19, n^{o} 01, pp. 129-166, This paper is dedicated to J.-B. Hiriart-Urruty. Note on v3: to appear in ESAIM: COCV [*DOI :* 10.1051/cocv/2012002], http://hal.inria.fr/hal-00569010

International Conferences with Proceedings

[23] J. F. BONNANS. *Singular arcs in the optimal control of a parabolic equation*, in "13th European Control Conference", Zurich, Switzerland, July 2013, http://hal.inria.fr/hal-00845704

[24] J. F. BONNANS. *The shooting approach to optimal control problems*, in "IFAC International Workshop on Adaptation and Learning in Control and Signal Processing", Caen, France, F. GIRI, V. VAN ASSCHE (editors), IFAC, July 2013, pp. 281-292 [*DOI :* 10.3182/20130703-3-FR-4038.00158], http://hal.inria.fr/hal-00830896

[25] B. BONNARD, M. CLAEYS, O. COTS, P. MARTINON. *Comparison of Numerical Methods in the Contrast Imaging Problem in NMR*, in "52nd IEEE Conference on Decision and Control", Firenze, Italy, December 2013, http://hal.inria.fr/hal-00800436

[26] A. FESTA, R. VINTER. *A decomposition technique for pursuit evasion games with many pursuers*, in "52nd IEEE Control and Decision Conference (CDC)", Florence, Italy, 2013, 7 p. , http://hal.inria.fr/hal-00916864

[27] R. MUÑOZ-TAMAYO, P. MARTINON, G. BOUGARAN, F. MAIRET, O. BERNARD. *Design of Optimal Experiments for Parameter Estimation of Microalgae Growth Models*, in "Computer Applied to Biotechnology", Mumbai, India, December 2013, http://hal.inria.fr/hal-00856749

Scientific Books (or Scientific Book chapters)

[28] Z. RAO, H. ZIDANI. *Hamilton-Jacobi-Bellman Equations on Multi-Domains*, in "Control and Optimization with PDE Constraints", K. BREDIES, C. CLASON, K. KUNISCH, G. VON WINCKEL (editors), International Series of Numerical Mathematics, Springer, May 2013, vol. 164, pp. 93–116 [*DOI :* 10.1007/978-3-0348-0631-2_6], http://hal.inria.fr/hal-00803108

Research Reports

[29] A. AFTALION, J. F. BONNANS. , *Optimization of running strategies based on anaerobic energy and variations of velocity*, Inria, August 2013, n^{o} RR-8344, 22 p. , http://hal.inria.fr/hal-00851182

[30] T. BAYEN, F. MAIRET, P. MARTINON, M. SEBBAH. , *Optimizing the anaerobic digestion of microalgae in a coupled process*, September 2013, http://hal.inria.fr/hal-00860570

[31] J. F. BONNANS, X. DUPUIS, L. PFEIFFER. , *Second-order necessary conditions in Pontryagin form for optimal control problems*, Inria, May 2013, n^o RR-8306, 37 p. , http://hal.inria.fr/hal-00825273

[32] J. F. BONNANS, X. DUPUIS, L. PFEIFFER. , *Second-order sufficient conditions for strong solutions to optimal control problems*, Inria, May 2013, n^o RR-8307, 24 p. , http://hal.inria.fr/hal-00825260

[33] X. DUPUIS. , *Optimal control of leukemic cell population dynamics*, Inria, August 2013, n^o RR-8356, 25 p. , http://hal.inria.fr/hal-00858208

Other Publications

[34] T. BAYEN, J. F. BONNANS, F. J. SILVA. , *Characterization of local quadratic growth for strong minima in the optimal control of semi-linear elliptic equations*, 2013, http://hal.inria.fr/hal-00913725

[35] O. BOKANOWSKI, A. PICARELLI, H. ZIDANI. , *Dynamic Programming and Error Estimates for Stochastic Control Problems with Maximum Cost*, January 2014, http://hal.inria.fr/hal-00931025

[36] B. BONNARD, M. CLAEYS, O. COTS, P. MARTINON. , *Geometric and numerical methods in the contrast imaging problem in nuclear magnetic resonance*, September 2013, http://hal.inria.fr/hal-00867753

[37] P. CARDALIAGUET, P. J. GRABER. , *Mean field games systems of first order*, January 2014, http://hal.inria.fr/hal-00925905

[38] N. FORCADEL, Z. RAO. , *Singular perturbation of optimal control problems on multi-domains*, April 2013, http://hal.inria.fr/hal-00812846

[39] L. GIRALDI, P. MARTINON, M. ZOPPELLO. , *Controllability and Optimal Strokes for N-link Micro-swimmer*, 2013, http://hal.inria.fr/hal-00798363

[40] P. J. GRABER. , *Optimal control of first-order Hamilton-Jacobi equations with linearly bounded Hamiltonian*, 2013, http://hal.inria.fr/hal-00871964

[41] L. GRÜNE, H. ZIDANI. , *Zubov's equation for state-constrained perturbed nonlinear systems*, 2013, http://hal.inria.fr/hal-00931028

[42] A. KROENER, K. KUNISCH, H. ZIDANI. , *Optimal feedback control of undamped wave equations by solving a HJB equation*, 2013, http://hal.inria.fr/hal-00924089

[43] Z. RAO, A. SICONOLFI, H. ZIDANI. , *Transmission conditions on interfaces for Hamilton-Jacobi-Bellman equations*, May 2013, http://hal.inria.fr/hal-00820273

References in notes

[44] R. ABGRALL. *Numerical discretization of boundary conditions for first order Hamilton-Jacobi equations*, in "SIAM J. Numerical Analysis", 2003, vol. 41, n^o 6, pp. 2233–2261

[45] R. ABGRALL, S. AUGOULA. *High order numerical discretization for Hamilton-Jacobi equations on triangular meshes*, in "J. Scientific Computing", 2000, vol. 15, n^{o} 2, pp. 197–229

[46] M. S. ARONNA, J. F. BONNANS, P. MARTINON. *A Shooting Algorithm for Optimal Control Problems with Singular Arcs*, in "Journal of Optimization Theory and Applications", 2013, Inria Report RR-7763, 2011

[47] J. AUBIN, H. FRANKOWSKA. , *Set-valued analysis*, Birkhäuser, Boston, 1990

[48] M. BARDI, I. CAPUZZO-DOLCETTA. , *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Systems and Control: Foundations and Applications, Birkhäuser, Boston, 1997

[49] G. BARLES. , *Solutions de viscosité des équations de Hamilton-Jacobi*, Mathématiques et Applications, Springer, Paris, 1994, vol. 17

[50] G. BARLES, E. JAKOBSEN. *Error bounds for monotone approximation schemes for Hamilton-Jacobi-Bellman equations*, in "SIAM J. Numerical Analysis", 2005, vol. 43, n^{o} 2, pp. 540–558

[51] G. BLISS. , *Lectures on the Calculus of Variations*, University of Chicago PressChicago, Illinois, 1946

[52] J. F. BONNANS, A. HERMANT. *Well-Posedness of the Shooting Algorithm for State Constrained Optimal Control Problems with a Single Constraint and Control*, in "SIAM J. Control Optimization", 2007, vol. 46, n^{o} 4, pp. 1398–1430

[53] J. F. BONNANS, A. HERMANT. *Second-order Analysis for Optimal Control Problems with Pure State Constraints and Mixed Control-State Constraints*, in "Annales de l'Institut Henri Poincaré. Analyse non linéaire", 2009, vol. 26, n^{o} 2, pp. 561-598

[54] J. F. BONNANS, J. LAURENT-VARIN. *Computation of order conditions for symplectic partitioned Runge-Kutta schemes with application to optimal control*, in "Numerische Mathematik", 2006, vol. 103, n^{o} 1, pp. 1–10

[55] F. J. BONNANS, P. MARTINON, V. GRÉLARD. , *Bocop - A collection of examples*, Inria, August 2012, n^{o} RR-8053, http://hal.inria.fr/hal-00726992

[56] J. F. BONNANS, E. OTTENWAELTER, H. ZIDANI. *Numerical schemes for the two dimensional second-order HJB equation*, in "ESAIM: M2AN", 2004, vol. 38, pp. 723-735

[57] J. F. BONNANS, H. ZIDANI. *Consistency of generalized finite difference schemes for the stochastic HJB equation*, in "SIAM J. Numerical Analysis", 2003, vol. 41, pp. 1008-1021

[58] A. E. BRYSON, Y.-C. HO. , *Applied optimal control*, Hemisphere PublishingNew-York, 1975

[59] F. CLARKE. *A new approach to Lagrange multipliers*, in "Mathematics of Operations Research", 1976, vol. 2, pp. 165-174

[60] M. CRANDALL, L. EVANS, P. LIONS. *Some properties of viscosity solutions of Hamilton-Jacobi equations*, in "Trans. Amer. Math. Soc", 1984, vol. 282, pp. 487-502

[61] M. CRANDALL, P. LIONS. *Viscosity solutions of Hamilton Jacobi equations*, in "Bull. American Mathematical Society", 1983, vol. 277, pp. 1–42

[62] M. CRANDALL, P. LIONS. *Two approximations of solutions of Hamilton-Jacobi equations*, in "Mathematics of Computation", 1984, vol. 43, pp. 1–19

[63] B. DESPRÉS, F. LAGOUTIÈRE. *Contact discontinuity capturing schemes for linear advection and compressible gas dynamics*, in "J. Sci. Comput.", 2001, vol. 16, pp. 479-524

[64] B. DESPRÉS, F. LAGOUTIÈRE. *A non-linear anti-diffusive scheme for the linear advection equation*, in "C. R. Acad. Sci. Paris, Série I, Analyse numérique", 1999, vol. 328, pp. 939-944

[65] A. DONTCHEV, W. HAGER, V. VELIOV. *Second-order Runge-Kutta approximations in control constrained optimal control*, in "SIAM Journal on Numerical Analysis", 2000, vol. 38, pp. 202–226

[66] I. EKELAND. *Nonconvex minimization problems*, in "Bulletin of the American Mathematical Society", 1979, vol. 1(New series), pp. 443-474

[67] M. FALCONE, R. FERRETTI. *Semi-Lagrangian schemes for Hamilton-Jacobi equations, discrete representation formulae and Godunov methods*, in "Journal of Computational Physics", 2002, vol. 175, pp. 559–575

[68] M. FALCONE, R. FERRETTI. *Convergence analysis for a class of high-order semi-Lagrangian advection schemes*, in "SIAM J. Numer. Anal.", 1998, vol. 35, n^o 3, pp. 909–940

[69] J. GERGAUD, P. MARTINON. *Using switching detection and variational equations for the shooting method*, in "Optimal Control Applications and Methods", 2007, vol. 28, n^o 2, pp. 95–116

[70] W. HAGER. *Runge-Kutta methods in optimal control and the transformed adjoint system*, in "Numerische Mathematik", 2000, vol. 87, n^o 2, pp. 247–282

[71] E. HARTEN. *ENO schemes with subcell resolution*, in "J. Computational Physics", 1989, vol. 83, pp. 148–184

[72] C. HU, C.-W. SHU. *A discontinuous Galerkin finite element method for Hamilton-Jacobi equations*, in "SIAM J. on Scientific Computing", 1999, vol. 21, n^o 2, pp. 666–690

[73] A. IOFFE, V. TIHOMIROV. , *Theory of Extremal Problems*, North-Holland Publishing Company, Amsterdam, 1979, Russian Edition: Nauka, Moscow, 1974

[74] J. B. KELLER. *A Theory of Competitive Running*, in "Physics Today", 1973, vol. 26, n^o 9, pp. 42–47

[75] J. B. KELLER. *Optimal velocity in a race*, in "Amer. Math. Monthly", 1974, vol. 81, pp. 474–480

[76] N. KRYLOV. *On the rate of convergence of finite-difference approximations for Bellman's equations with variable coefficients*, in "Probability Theory and Related Fields", 2000, vol. 117, pp. 1–16

[77] H. KUSHNER, P. DUPUIS. , *Numerical methods for stochastic control problems in continuous time*, Applications of mathematics, Springer, New York, 2001, vol. 24, Second edition

[78] J.-L. LAGRANGE. , *Mécanique analytique*, Paris, 1788, reprinted by J. Gabay, 1989

[79] E. LEE, L. MARKUS. , *Foundations of optimal control theory*, John Wiley, New York, 1967

[80] G. LEITMANN. , *An introduction to optimal control*, Mc Graw Hill, New York, 1966

[81] K. MALANOWSKI, C. BÜSKENS, H. MAURER. *Convergence of approximations to nonlinear optimal control problems*, in "Mathematical programming with data perturbations", New York, Lecture Notes in Pure and Appl. Math., Dekker, 1998, vol. 195, pp. 253–284

[82] S. OSHER, C.-W. SHU. *High essentially nonoscillatory schemes for Hamilton-Jacobi equations*, in "SIAM J. Numer. Anal.", 1991, vol. 28, n^{o} 4, pp. 907-922

[83] H. PHAM. , *Optimisation et Contrôle Stochastique Appliqués à la Finance*, Springer Verlag, 2007, n^{o} 61

[84] L. PONTRYAGIN, V. BOLTYANSKI, R. GAMKRELIDZE, E. MICHTCHENKO. , *The Mathematical Theory of Optimal Processes*, Wiley Interscience, New York, 1962

[85] P. ROE. *Some contributions to the modelling of discontinuous flows*, in "Lectures in Applied Mathematics", 1985, vol. 22, pp. 163–193

[86] L. YOUNG. , *Lectures on the calculus of variations and optimal control theory*, W. B. Saunders Co.Philadelphia, 1969

[87] B. ØKSENDAL. , *Stochastic differential equations*, Universitext, Sixth, Springer-VerlagBerlin, 2003