Activity Report 2014

Team MOKAPLAN

Méthodes numériques pour le problème de Monge-Kantorovich et Applications en sciences sociales

RESEARCH CENTER
Paris - Rocquencourt

THEME
Numerical schemes and simulations

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Team MOKAPLAN

Keywords: Socio-economic Models, Optimization, Numerical Methods

Creation of the Team: 2013 January 01.

1. Members

Research Scientists

Jean-David Benamou [Team leader, Inria, Senior Researcher, HdR]

Vincent Duval [Inria, from Sep 2014]

Faculty Member

Guillaume Carlier [Univ. Paris IX, Professor, HdR]

Engineers

Simon Legrand [Inria, from Nov 2014]

Francis Collino [Senior engineer,granted by ANR ISOTACE project, consultant]

Christophe Duquesne [Senior engineer,granted by ANR ISOTACE project, consultant]

PhD Student

Luca Nenna [Inria]

Post-Doctoral Fellow

Xavier Dupuis [Inria]

Administrative Assistant

Martine Verneuille [Inria]

Other

Aude Genevay [Inria, Stage M2, from Jun 2014 until Sep 2014]

2. Overall Objectives

2.1. Introduction

Over the last twenty years, Optimal Mass Transportation has played a major role in PDEs, geometry, functional inequalities as well as in modelling and applied fields such as fluid mechanics, image processing and economics. This trend shows no sign of slowing and the field is still extremely active. However, the numerics remain underdeveloped, but recent progress in this new field of numerical Optimal Mass Transportation raise hope for significant advances in numerical simulations.

Mokaplan objectives are to design, develop and implement these new algorithms with and emphasis on economic applications.

3. Research Program

3.1. Context

Optimal Mass Transportation is a mathematical research topic which started two centuries ago with Monge's work on "des remblais et déblais". This engineering problem consists in minimizing the transport cost between two given mass densities. In the 40's, Kantorovich [64] solved the dual problem and interpreted it as an economic equilibrium. The *Monge-Kantorovich* problem became a specialized research topic in optimization and Kantorovich obtained the 1975 Nobel prize in economics for his contributions to resource allocations problems. Following the seminal discoveries of Brenier in the 90's [35], Optimal Transportation has received renewed attention from mathematical analysts and the Fields Medal awarded in 2010 to C. Villani, who gave important contributions to Optimal Transportation and wrote the modern reference monograph [84], arrived at a culminating moment for this theory. Optimal Mass Transportation is today a mature area of mathematical analysis with a constantly growing range of applications (see below).

In the modern Optimal Mass Transportation problem, we are given two probability measures or "mass" densities : $d\rho_i(x_i)(=\rho_i(x_i)\,dx_i),\ i=0,1$ such that $\rho_i \geq 0,\ \int_{X_0}\rho_0(x_0)dx_0 = \int_{X_1}\rho_1(x_1)dx_1 = 1$, $X_i \subset R^n$. They are often referred to, respectively, source and target densities, support or spaces. The problem is the minimization of a *transportation cost*, $\mathcal{I}(M) = \int_{X_0} c(x, M(x))\,\rho_0(x)dx$ where c is a displacement *ground cost*, over all *volume preserving maps* $M \in \mathcal{M}$ $\mathcal{M} = \{M : X_0 \to X_1,\ M_{\#}d\rho_0 = d\rho_1\}$. Assuming that M is a diffeomorphism, this is equivalent to the *Jacobian equation* $det(DM(x))\rho_1(M(x)) = \rho_0(x)$. Most of the modern Optimal Mass Transportation theory has been developed for the Euclidean distance squared cost $c(x,y) = \|x-y)\|^2$ while the historic monge cost was the simple distance $c(x,y) = \|x-y\|$.
In the Euclidean distance squared ground cost, the problem is well posed and in the seminal work of Brenier [36], the optimal map is characterized as the gradient of a convex potential ϕ^* : $\mathcal{I}(\nabla\phi^*(x)) = \min_{M\in\mathcal{M}} \mathcal{I}(M)$. A formal substitution in the Jacobian equation gives the Monge-Ampère equation $det(D^2\phi^*)\rho_1(\nabla\phi^*(x)) = \rho_0(x)$ complemented by the *second boundary value* condition $\nabla\phi^*(X_0) \subset X_1$. Caffarelli [41] used this result to extend the regularity theory for the Monge-Ampère equation. He noticed in particular that Optimal Mass Transportation solutions, now called *Brenier solutions*, may have discontinuous gradients when the target density support X_1 is non convex and are therefore weaker than the Monge-Ampère potentials associated to Alexandrov measures (see [60] for a review of the different notions of Monge-Ampère solutions). The value function $\sqrt{\mathcal{I}(\nabla\phi^*)}$ is also known to be the *Wasserstein distance* $W_2(\rho_0,\rho_1)$ on the space of probability densities, see [84]. The *Computational Fluid Dynamic* formulation proposed by Brenier and Benamou in [2] introduces a time extension of the domain and leads to a convex but non smooth optimization problem : $\mathcal{I}(\nabla\phi^*) = \min_{(\rho,V)\in\mathcal{C}} \displaystyle\int_0^1\int_X \frac{1}{2}\rho(t,x)\,\|V(t,x)\|^2\,dx\,dt$. with constraints : $\mathcal{C} = \{(\rho,V),$ s.t $\partial_t\rho + div(\rho V) = 0,\ \rho(\{0,1\},.) = \rho_{\{0,1\}}(.)\}$. The time curves $t \to \rho(t,.)$ are geodesics between ρ_0 and ρ_1 for the Wasserstein distance. This formulation is a limit case of *Mean Fields games* [65], a large class of economic models introduced by Lasry and Lions. The Wasserstein distance and its connection to Optimal Mass Transportation also appears in the construction of semi-discrete Gradient Flows. This notion known as *JKO gradient flows* after its authors in [62] is a popular tool to study non-linear diffusion equations : the implicit Euler scheme $\rho^{dt}_{k+1} = argmin_{\rho(.)}\,F(\rho(.)) + \frac{1}{2\,dt}W_2(\rho(.),\rho^{dt}_k)^2$ can be shown to converge $\rho^{dt}_k(.)) \to \rho^*(t,.)$ as $dt \to 0$ to the solution of the non linear continuity equation $\partial_t\rho^* + div(\rho^*\,\nabla(-\frac{\partial F}{\partial\rho}(\rho^*))) = 0,\ \rho^*(0,.) = \rho^{dt}_0(.)$. The prototypical example is given by $F(\rho) = \int_X \rho(x)\,\log(\rho(x)) + \rho(x)\,V(x)\,dx$ which corresponds to the classical Fokker-Planck equation. Extensions of the ground cost c have been actively studied recently, some are mentioned in the application section. Technical results culminating with the *Ma-Trudinger-Wang* condition [68] which gives necessary condition on c for the regularity of the solution of the Optimal Mass Transportation problem. More recently attention has risen on multi marginal Optimal Mass Transportation [59] and has been systematically studied in [76] [79] [77] [78]. The data consists in an arbitrary (and even infinite) number N of densities (the marginals) and the ground cost is defined on a product space $c(x_0, x_1, ..., x_{n-1})$ of the same dimension. Several interesting applications belong to this class of models (see below).
Our focus is on numerical methods in Optimal Mass Transportation and applications. The simplest way to build a numerical method is to consider sum of dirac masses $\rho_0 = \sum_{i=1}^N \delta_{A_i}$ $\rho_1 = \sum_{j=1}^N \delta_{B_j}$. In that case the Optimal Mass Transportation problem reduces to combinatorial optimisation *assigment problem* between the points $\{A_i\}$s and $\{B_i\}$s : $\min_{\sigma\in Permut(1,N)} \frac{1}{N}\sum_{i=1}^N C_{i,\sigma(i)} C_{i,j} = \|A_i - B_j\|^2$. The complexity of the best (Hungarian or Auction) algorithm, see [33] for example, is $O(N^{\frac{5}{2}})$. An interesting variant is obtained when only the target measure is discrete. For instance $X_0 = \{\|x\| < 1\},\ \rho_0 = \frac{1}{|X_0|}$ $\rho_1 = \frac{1}{N}\sum_{j=1}^N \delta_{y_j}$. It corresponds to the notion of Pogorelov solutions of the Monge-Ampère equation [80] and is also linked to Minkowski problem [31]. The optimal map is piecewise constant and the slopes are known. More precisely there exists N polygonal cells C_j such that $X_0 = \cup_j C_j$, $|C_j| = \frac{1}{N}$ and $\nabla\phi^*|_{C_j} = y_j$. Pogorelov proposed a constructive algorithm to build these solutions which has been refined and extended in particular in [50] [74] [72] [71]. The complexity is still not linear : $O(N^2 log N)$.

For general densities data, the original optimization problem is not tractable because of the volume preserving constraint on the map. Kantorovich dual formulation is a linear program but with a large number of constraints set over the product of the source and target space $X_0 \times X_1$. The CFD formulation [2]. preserves the convexity of the objective function and transforms the volume preserving constraint into a linear continuity equation (using a change of variable). We obtained a convex but non smooth optimization problem solved using an Augmented Lagrangian method [53], as originally proposed in [2]. It has been reinterpreted recently in the framework of proximal algorithms [75]. This approach is robust and versatile and has been reimplemented many times. It remains a first order optimization method and converges slowly. The cost is also increased by the additional artificial time dimension. An empirical complexity is $O(N^3 LogN)$ where N is the space discretization of the density. Several variants and extension of these methods have been implemented, in particular in [39] [30]. It is the only provably convergent method to compute Brenier (non C^1) solutions.
When interested in slightly more regular solutions which correspond to the assumption that the target support is convex, the recent *wide stencil* monotone finite difference scheme for the Monge-Ampère equation [55] can be adapted to the Optimal Mass Transportation problem. This is the topic of [7]. This approach is extremely fast as a Newton algorithm can be used to solve the discrete system. Numerical studies confirm this with a linear empirical complexity.

For other costs, JKO schemes, multi marginal extensions, partial transport ... efficient numerical methods are to be invented.

4. Application Domains

4.1. Continuous models in economics

- As already mentioned the CFD formulation is a limit case of simple variational Mean-Field Games (MFG) [65]. MFG is a new branch of game theory recently developed by J-M. Lasry and P-L. Lions. MFG models aim at describing the limiting behavior of stochastic differential games when the number of players tends to infinity. They are specifically designed to model economic problems where a large number of similar interacting agents try to maximize/minimize a utility/cost function which takes into account global but partial information on the game. The players in these models are individually insignificant but they collectively have a significant impact on the cost of the other players. Dynamic MFG models often lead to a system of PDEs which consists of a backward Hamilton-Jacobi Bellman equation for a value function coupled with a forward Fokker-Planck equation describing the space-time evolution of the density of agents.
- In microeconomics, the *principal-agent problem* [83] with adverse selection plays a distinguished role in the literature on asymmetric information and contract theory (with important contributions from several Nobel prizes such as Mirrlees, Myerson, Spence or Tirole) and it has many important applications in optimal taxation, insurance, nonlinear pricing. The problem can be reduced to the maximization of an integral functional subject to a convexity constraint This is an unusual calculus of variations problem and the optimal price can only be computed numerically. Recently, following a reformulation of Carlier [12], convexity/well-posedness results of McCann, Figalli and Kim [52], connected to optimal transport theory, showed that there is some hope to numerically solve the problem for general utility functions.
- In [9] a class of games are considered with a continuum of players for which Cournot-Nash equilibria can be obtained by the minimisation of some cost, related to optimal transport. This cost is not convex in the usual sense in general but it turns out to have hidden strict convexity properties in many relevant cases. This enables us to obtain new uniqueness results and a characterisation of equilibria in terms of some partial differential equations, a simple numerical scheme in dimension one as well as an analysis of the inefficiency of equilibria. The mathematical problem has the structure of one step of the JKO gradient flow method.

- Many relevant markets are markets of indivisible goods characterized by a certain quality: houses, jobs, marriages... On the theoretical side, recent papers by Ekeland, McCann, Chiappori [45] showed that finding equilibria in such markets is equivalent to solving a certain optimal transport problem (where the cost function depends on the sellers and buyers preferences). On the empirical side, this allows for trying to recover information on the preferences from observed matching; this is an inverse problem as in a recent work of Galichon and Salanié [57] [58] Interestingly, these problems naturally lead to numerically challenging variants of the Monge-Kantorovich problem: the multi-marginal OT problem and the entropic approximation of the Monge-Kantorovich problem (which is actually due to Schrödinger in the early 30's).

4.2. Finance

The Skorohod embedding problem (SEP) consists in finding a martingale interpolation between two probability measures. When a particular stochastic ordering between the two measures is given, Galichon et al [56] have shown that a very natural variational formulation could be given to a class of problems that includes the SEP. This formulation is related to the CFD formulation of the OT problem [2] and has applications to *model-free bounds of derivative prices in Finance*. It can also be interpreted as a a multi marginal Optimal Mass Transportation with infinitely many marginals [78].

4.3. Congested Crowd motion

The volume preserving property appears naturally in this context where motion is constrained by the density of player.

- Optimal Mass Transportation and MFG theories can be an extremely powerful tool to attack some of these problems arising from spatial economics or to design new ones. For instance, various urban/traffic planning models have been proposed by Buttazzo, Santambrogio, Carlier ([10] [40] [32]) in recent years.
- Many models from PDEs and fluid mechanics have been used to give a description of *people or vehicles moving in a congested environment*. These models have to be classified according to the dimension (1D model are mostly used for cars on traffic networks, while 2D models are most suitable for pedestrians), to the congestion effects ("soft" congestion standing for the phenomenon where high densities slow down the movement, "hard" congestion for the sudden effects when contacts occur, or a certain threshold is attained), and to the possible rationality of the agents Maury et al [69] recently developed a theory for 2D hard congestion models without rationality, first in a discrete and then in a continuous framework. This model produces a PDE that is difficult to attack with usual PDE methods, but has been successfully studied via Optimal Mass Transportation techniques again related to the JKO gradient flow paradigm.

4.4. Astrophysics

In [54] and [37], the authors show that the deterministic past history of the Universe can be uniquely reconstructed from the knowledge of the present mass density field, the latter being inferred from the 3D distribution of luminous matter, assumed to be tracing the distribution of dark matter up to a known bias. Reconstruction ceases to be unique below those scales – a few Mpc – where multi-streaming becomes significant. Above 6 Mpc/h we propose and implement an effective Monge-Ampere-Kantorovich method of unique reconstruction. At such scales the Zel'dovich approximation is well satisfied and reconstruction becomes an instance of optimal mass transportation. After discretization into N point masses one obtains an assignment problem that can be handled by effective algorithms with not more than cubic time complexity in N and reasonable CPU time requirements. Testing against N-body cosmological simulations gives over 60% of exactly reconstructed points.

4.5. Image Processing and inverse problems

The Wasserstein distance between densities is the value function of the Optimal Mass Transportation problem. This distance may be considered to have "orthogonal" properties to the widely used least square distance. It is for instance quadratic with respect to dilations and translation. On the other hand it is not very sensitive to rigid transformations, [75] is an attempt at generalizing the CFD formulation in this context. The Wasserstein distance is an interesting tool for applications where distances between signals and in particular oscillatory signals need to to computed, this is assuming one understands how to transform the information into positive densities.

- Tannenbaum and co-authors have designed several variants of the CFD numerical method and applied it to warping, morphing and registration (using the Optimal Mass Transportation map) problems in medical imaging. [86] [30]
- Gabriel Peyre and co-authors [82] have proposed an easier to compute relaxation of the Wasserstein distance (the sliced Wasserstein distance) and applied it to two image processing problems: color transfer and texture mixing.
- Froese Engquist [51] use a Monge-Ampère Solver to compute the Wasserstein distance between synthetic 2D Seismic signals (After some transformations). Applications to waveform inversion and registration are discussed and simple numerical examples are presented.

4.6. Meteorology and Fluid models

In, [34] Brenier reviews in a unified framework the connection between optimal transport theory and classical convection theory for geophysical flows. Inspired by the numerical model proposed in [30], the starting point is a generalization of the Darcy-Boussinesq equations, which is a degenerate version of the Navier-Stokes-Boussinesq (NSB) equations. In a unified framework, he relates different variants of the NSB equations (in particular what he calls the generalized hydrostatic-Boussinesq equations) to various models involving optimal transport and the related Monge-Ampère equation. This includes the 2D semi-geostrophic equations [61] [49] [48] [4] [67] and some fully nonlinear versions of the so-called high-field limit of the Vlasov-Poisson system [73] and of the Keller-Segel system for chemotaxis [63] [44] .

4.7. Mesh motion/Lagragian methods

The necessity to preserve areas/volumes is a intrinsic feature of mesh deformations more generally Lagrangian numerical methods. Numerical method of Optimal Mass Transportation which preserve some notions of convexity and as a consequence the monotonicity of the computed transport maps can play a role in this context, see for instance [43] [46] [66].

4.8. Density Functionnal Theory (DFT)

The precise modeling of electron correlations continues to constitute the major obstacle in developing high-accuracy, low-cost methods for electronic structure computations in molecules and solids. The article [47] sheds a new light on the longstanding problem of how to accurately incorporate electron correlation into DFT, by deriving and analyzing the semiclassical limit of the exact Hohenberg-Kohn functional with the single-particle density ρ held fixed. In this limit, in the case of two electrons, the exact functional reduces to a very interesting functional that depends on an optimal transport map M associated with a given density ρ. The limit problem is known in the DFT literature with the optimal transport map being called a correlation function or a co-motion function , but it has not been rigorously derived, and it appears that it has not previously been interpreted as an optimal transport problem. The article [47] thereby links for the first time DFT, which is a large and very active research area in physics and chemistry, to optimal transportation theory with a Coulombian repulsive cost. Numerics are still widely open [38].

5. New Software and Platforms

5.1. ALG2 for Monge Mean-Field Games, Monge problem and Variational problems under divergence constraint

5.1.1. Platforms

A generalisation of the ALG2 algorithm [53] corresponding to the paper [18] ha been implemented in FreeFem++. The scripts and numerical simulations are available at https://team.inria.fr/mokaplan/augmented-lagrangian-simulations/.

We still plan to implement a parallel version on Rocquencourt Inria cluster. We are waiting for FreeFem to be installed on the cluster.

5.2. Mokabajour

5.2.1. Platforms

Following the pioneering work of Caffarelli and Oliker [42], Wang [85] has shown that the inverse problem of freeforming a *convex* reflector which sends a prescribed source to a target intensity is a particular instance of Optimal Mass Transportation. The method developed in [7] has been used by researchers of TU Eindhoven in collaboration with Philips Lightning Labs to compute reflectors [81] in a simplified setting. The industrial motivation is the automatic design of reflector given prescribed source and target illuminance. From the mathematical point of view there is a hierarchy of Optimal Mass Transportation reflector and lenses problems and only the simplest "far field" one can be solved with state of the art Monge-Ampère solvers. We will adapt the Monge-Ampère solvers and also attempt to build real optimized reflector prototypes. We plan on investigating the more complicated near field models and design numerical methods. Finally Monge-Ampère based Optimal Mass Transportation solvers will be made available. This could be used for example in Mesh adaptation.

The web site is under construction https://project.inria.fr/mokabajour/, preliminary results are available.

This ADT (Simon Legrand) on the numerical free forming of specular reflectors started in december. We implement different types of MA solvers in collaboration with Quentin Mérigot (CEREMADE), Boris Thibert (LJK Grenoble) and Vincent Duval. See https://project.inria.fr/mokabajour/.

6. New Results

6.1. Highlights of the Year

All of the new results below are important break through and most of them non-incremental research.

Mokaplan has extended its collaborations to several researchers at Ceremade and is under review to become a project team.

6.2. Iterative Bregman Projections for Regularized Transportation Problems

Benamou, Jean-David and Carlier, Guillaume and Cuturi, Marco and Nenna, Luca and Peyré, Gabriel

[19]

We provide a general numerical framework to approximate solutions to linear programs related to optimal transport. The general idea is to introduce an entropic regularization of the initial linear program. This regularized problem corresponds to a Kullback-Leibler Bregman di-vergence projection of a vector (representing some initial joint distribution) on the polytope of constraints. We show that for many problems related to optimal transport, the set of linear constraints can be split in an intersection of a few simple constraints, for which the projections can be computed in closed form. This allows us to make use of iterative Bregman projections (when there are only equality constraints) or more generally Bregman-Dykstra iterations (when inequality constraints are involved). We illustrate the usefulness of this approach to several variational problems related to optimal transport: barycenters for the optimal trans-port metric, tomographic reconstruction, multi-marginal optimal trans-port and in particular its application to Brenier's relaxed solutions of in-compressible Euler equations, partial unbalanced optimal transport and optimal transport with capacity constraints.
The extension of the method to the Principal Agent problem, Density Functional theory and Transport under martingal constraint is under way.

6.3. A viscosity framework for computing Pogorelov solutions of the Monge-Ampere equation

Benamou, Jean-David and Froese, Brittany D.

[21]

We consider the Monge-Kantorovich optimal transportation problem between two measures, one of which is a weighted sum of Diracs. This problem is traditionally solved using expensive geometric methods. It can also be reformulated as an elliptic partial differential equation known as the Monge-Ampere equation. However, existing numerical methods for this non-linear PDE require the measures to have finite density. We introduce a new formulation that couples the viscosity and Aleksandrov solution definitions and show that it is equivalent to the original problem. Moreover, we describe a local reformulation of the subgradient measure at the Diracs, which makes use of one-sided directional derivatives. This leads to a consistent, monotone discretisation of the equation. Computational results demonstrate the correctness of this scheme when methods designed for conventional viscosity solutions fail.
The method offers a new insight into the duality between Aleksandrov and Brenier solutions of the Monge Ampère equations. We still work on the viscosity existence/uniqueness convergence of sheme theory.

6.4. Discretization of functionals involving the Monge-Ampère operator

Benamou, Jean-David and Carlier, Guillaume and Mérigot, Quentin and Oudet, Edouard

[26]

Gradient flows in the Wasserstein space have become a powerful tool in the analysis of diffusion equations, following the seminal work of Jordan, Kinderlehrer and Otto (JKO). The numerical applications of this formulation have been limited by the difficulty to compute the Wasserstein distance in dimension larger than 2. One step of the JKO scheme is equivalent to a variational problem on the space of convex functions, which involves the Monge-Ampère operator. Convexity constraints are notably difficult to handle numerically, but in our setting the internal energy plays the role of a barrier for these constraints. This enables us to introduce a consistent discretization, which inherits convexity properties of the continuous variational problem. We show the effectiveness of our approach on nonlinear diffusion and crowd-motion models.

6.5. Augmented Lagrangian methods for transport optimization, Mean-Field Games and degenerate PDEs

Benamou, Jean-David and Carlier, Guillaume

[18]

Many problems from mass transport can be reformulated as variational problems under a prescribed divergence constraint (static problems) or subject to a time dependent continuity equation which again can also be formulated as a divergence constraint but in time and space. The variational class of Mean-Field Games introduced by Lasry and Lions may also be interpreted as a generalisation of the time-dependent optimal transport problem. Following Benamou and Brenier, we show that augmented Lagrangian methods are well-suited to treat convex but nonsmooth problems. It includes in particular Monge historic optimal transport problem. A Finite Element discretization and implementation of the method is used to provide numerical simulations and a convergence study.
We have good hopes to use this method to many non-linear diffusion equations through the use of JKO gradient schemes.

6.6. Discretization of functionals involving the Monge-Ampère operator

Benamou, Jean-David and Collino, Francis and Mirebeau, Jean-Marie

[20]

We introduce a novel discretization of the Monge-Ampere operator, simultaneously consistent and degenerate elliptic, hence accurate and robust in applications. These properties are achieved by exploiting the arithmetic structure of the discrete domain, assumed to be a two dimensional cartesian grid. The construction of our scheme is simple, but its analysis relies on original tools seldom encountered in numerical analysis, such as the geometry of two dimensional lattices, and an arithmetic structure called the Stern-Brocot tree. Numerical experiments illustrate the method's efficiency.

6.7. A Γ-Convergence Result for the Upper Bound Limit Analysis of Plates

Bleyer, Jérémy and Carlier, Guillaume and Duval, Vincent and Mirebeau, Jean-Marie and Peyré, Gabriel

[23]

Upper bound limit analysis allows one to evaluate directly the ultimate load of structures without performing a cumbersome incremental analysis. In order to numerically apply this method to thin plates in bending, several authors have proposed to use various finite elements discretizations. We provide in this paper a mathematical analysis which ensures the convergence of the finite element method, even with finite elements with discontinuous derivatives such as the quadratic 6 node Lagrange triangles and the cubic Hermite triangles. More precisely, we prove the Gamma-convergence of the discretized problems towards the continuous limit analysis problem. Numerical results illustrate the relevance of this analysis for the yield design of both homogeneous and non-homogeneous materials.

6.8. Cournot-Nash equilibria

Carlier, Guillaume and Blanchet, Adrien

[24]

The notion of Nash equilibria plays a key role in the analysis of strategic interactions in the framework of N player games. Analysis of Nash equilibria is however a complex issue when the number of players is large. It is therefore natural to investigate the continuous limit as N tends to infinity and to investigate whether it corresponds to the notion of Cournot-Nash equilibria. In [9], this kind of convergence result is studied in a Wasserstein framework. In [BC1], we go one step further by giving a class of games with a continnum of players for which equilibria may be found as minimizers as a functional on measures which is very similar to the one-step JKO case, uniqueness results are the obtained from displacement convexity arguments. Finally, in [9] some situations which are non variational are considered and existence is obtained by methods combining fixed point arguments and optimal transport.

6.9. Principal Agent

Carlier, Guillaume, Benamou, Jean-David and Dupuis Xavier

The numerical resolution of principal Agent for a bilinear utility has been attacked and solved successfully in a series of recent papers see [70] and references therein.

A Bregman approach inspired by [6] has been developed for more general functions the paper is currently being written. It would be extremely useful as a complement to the theoretical analysis. A new semi-Discrete Geometric approach is also investigated where the method reduces to non-convex polynomial optimization.

6.10. Exact Support Recovery for Sparse Spikes Deconvolution

Duval, Vincent and Peyré, Gabriel

[17]

We study sparse spikes deconvolution over the space of measures. We focus our attention to the recovery properties of the support of the measure, i.e. the location of the Dirac masses. For non-degenerate sums of Diracs, we show that, when the signal-to-noise ratio is large enough, total variation regularization (which is the natural extension of the L1 norm of vectors to the setting of measures) recovers the exact same number of Diracs. We also show that both the locations and the heights of these Diracs converge toward those of the input measure when the noise drops to zero. The exact speed of convergence is governed by a specific dual certificate, which can be computed by solving a linear system. We draw connections between the support of the recovered measure on a continuous domain and on a discretized grid. We show that when the signal-to-noise level is large enough, the solution of the discretized problem is supported on pairs of Diracs which are neighbors of the Diracs of the input measure. This gives a precise description of the convergence of the solution of the discretized problem toward the solution of the continuous grid-free problem, as the grid size tends to zero.

7. Partnerships and Cooperations

7.1. National Initiatives

7.1.1. ANR

Jean-David Benamou is the coordinator of the ANR ISOTACE (Interacting Systems and Optimal Transportation, Applications to Computational Economics) ANR-12-MONU-0013 (2012-2016). The consortium explores new numerical methods in Optimal Transportation AND Mean Field Game theory with applications in Economics and congested crowd motion. Four extended seminars have been organized/co-organized by Mokaplan. Check https://project.inria.fr/isotace/news.
Christophe Duquesne (Aurigetech) is a software and mobility consultant hired on the ANR budget. He helps the consortium to develop its industrial partnerships.

7.2. International Initiatives

7.2.1. Inria Associate Teams

7.2.1.1. MOKALIEN

Title: Numerical Optimal Transportation in (Mathematical) Economics

International Partner (Institution - Laboratory - Researcher):

McGill University (CANADA)

Duration: 2014 - 2016

See also: https://team.inria.fr/mokaplan/mokalien/

The overall scientific goals is to develop numerical methods for large scale optimal transport and models based on optimal transport tools
see https://team.inria.fr/mokaplan/files/2014/09/MOKALIEN_Proposal_2013.pdf, section 2.

A few additional applications were suggested at our annual workshop in october
https://team.inria.fr/mokaplan/first-meeting-in-montreal-at-u-mcgill-october-20-24-2014/

7.3. International Research Visitors

7.3.1. Visits of International Scientists

Adam Oberman (U. Mc Gill) visited Mokaplan in June.

7.3.2. Visits to International Teams

7.3.2.1. Sabbatical programme

Guillaume Carlier in on sabbatical for the academic year (délégation CNRS at the UMI-CNRS 3069 PIMS at UVIC, Victoria, British Columbia, Canada). He is taking advantage of this full-research year to work on optimal transport methods for kinetic models for granular media (with M. Agueh and Reinhard Illner), Wasserstein barycenters and to continue to develop joint projects on numerical optimal transport with J.D. Benamou's MOKAPLAN team.

8. Dissemination

8.1. Promoting Scientific Activities

8.1.1. Scientific events organisation

8.1.1.1. Member of the organizing committee

Guillaume Carlier was organiser of the RICAM special semester on Calculus of Variation http://www.ricam.oeaw.ac.at/specsem/specsem2014/. This was a two weeks event. The first week was devoted to original minicourses on special research topics that are particularly active: Numerical mehods for optimal transport (J.-D. Benamou), Multi-marginal transport problems (L. de Pascale) and Gradient Flows (J.-A. Carrillo). The second week, a workshop was organized. This event gathered more than 50 participants with a majority of young researchers (more than 30).

8.1.2. Scientific events selection

8.1.2.1. Member of the conference program committee

Guillaume carlier is in the scientific committee for SMAI 2015.

8.1.2.2. Reviewer

Vincent Duval has reviewed several contributions for the *Scale Space and Variational Methods* SSVM 2015 conference.

8.1.3. Journal

8.1.3.1. Member of the editorial board

Guillaume carlier is member of the editorial Board of "Journal de l'Ecole Polytechnique" and co-editor of "Mathematics and Financial Economics".

8.1.3.2. Reviewer

Vincent Duval has reviewed several papers for the following jounals:

- SIIMS (SIAM Journal on Imaging Sciences)
- JMAA (Journal of Mathematical Analysis and Applications)
- IPol (Image Processing Online)
- JVCI (Journal of Visual Communication and Image Representation)

8.2. Teaching - Supervision - Juries

8.2.1. Teaching

In 2014, Guillaume Carlier gave an advanced course on Mean-Field Games (M2 EDP-MAD and Masef) and a master course (M1) on dynamic programming at Dauphine

Master : Guillaume Carlier, Mean-Field Games, M2 EDP-MAD , U. Paris- Dauphine.

Master : Guillaume Carlier, Dynamic Programming , M1 , U. Paris- Dauphine.

8.2.2. Supervision

PhD in progress: Quentin Denoyelle, “Analyse théorique et numérique de la super-résolution sans grille”, 2014, Gabriel Peyré and Vincent Duval.

PhD in progress : Roméo Hatchi , "Analyse mathématique de modèles de trafic congestionné", 2012, Guillaume Carlier.

PhD in progress : Maxime Laborde , " Dynamique des systèmes de particules en interaction, approche par flots de gradient et applications", 2013 , Guillaume Carlier

PhD in progress : Luca Nenna , "Méthodes numṕerique pour le transport optimal multimarge" , 2013, Jean-David Benamou et Guillaume Carlier.

PhD in progress: Quentin Denoyelle, “Analyse théorique et numérique de la super-résolution sans grille”, thèse commencée le 1er octobre 2014, supervised by Gabriel Peyré (main supervisor) and Vincent Duval (co-supervisor).

8.2.3. Juries

Guillaume carlier was in the Ph.D. committee of Serena Guarino (Pisa) and Miryana Grigorova (Paris 7).

9. Bibliography

Major publications by the team in recent years

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[7] J.-D. BENAMOU, B. D. FROESE, A. M. OBERMAN. *Numerical solution of the Optimal Transportation problem using the Monge–Ampère equation*, in "Journal of Computational Physics", March 2014, vol. 260, n^o 1, pp. 107–126 [*DOI :* 10.1016/J.JCP.2013.12.015], https://hal.inria.fr/hal-01115626

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Other Publications

[18] J.-D. BENAMOU, G. CARLIER. *Augmented Lagrangian methods for transport optimization, Mean-Field Games and degenerate PDEs*, October 2014, https://hal.inria.fr/hal-01073143

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