Activity Report 2016

Team TROPICAL

Tropical methods: structures, algorithms and interactions

Inria teams are typically groups of researchers working on the definition of a common project, and objectives, with the goal to arrive at the creation of a project-team. Such project-teams may include other partners (universities or research institutions).

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Team TROPICAL

Creation of the Team: 2016 January 01

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Computer Science and Digital Science:

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Other Research Topics and Application Domains:

1.1.10. - Mathematical biology
4.4. - Energy delivery
4.4.1. - Smart grids
9.9. - Risk management

1. Members

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Jean-Pierre Quadrat [Inria, Emiritus Senior Researcher, until Jun 2016, HDR]
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2. Overall Objectives

2.1. Introduction

The project develops tropical methods motivated by applications arising in decision theory (deterministic and stochastic optimal control, game theory, optimization and operations research), in the analysis or control of classes of dynamical systems (including timed discrete event systems and positive systems), in the verification of programs and systems, and in the development of numerical algorithms. Tropical algebra tools are used in interaction with various methods, coming from convex analysis, Hamilton–Jacobi partial differential equations, metric geometry, Perron-Frobenius and nonlinear fixed-point theories, combinatorics or algorithmic complexity. The emphasis of the project is on mathematical modelling and computational aspects.

The subtitle of the *Tropical* project, namely, "structures, algorithms, and interactions", refers to the spirit of our research, including a methodological component, computational aspects, and finally interactions with other scientific fields or real world applications, in particular through mathematical modelling.

2.2. Scientific context

Tropical algebra, geometry, and analysis have enjoyed spectacular development in recent years. Tropical structures initially arose to solve problems in performance evaluation of discrete event systems [66], combinatorial optimization [68], or automata theory [106]. They also arose in mathematical physics and asymptotic analysis [97], [94]. More recently, these structures have appeared in several areas of pure mathematics, in particular in the study of combinatorial aspects of algebraic geometry [87], [115], [108], [91], in algebraic combinatorics [82], and in arithmetics [72]. Also, further applications of tropical methods have appeared, including optimal control [100], program invariant computation [63] and timed systems verification [96], and zero-sum games [2].

The term 'tropical' generally refers to algebraic structures in which the laws originate from optimization processes. The prototypical tropical structure is the max-plus semifield, consisting of the real numbers, equipped with the maximum, thought of as an additive law, and the addition, thought of as a multiplicative law. Tropical objects appear as limits of classical objects along certain deformations ("log-limits sets" of Bergman, "Maslov dequantization", or "Viro deformation"). For this reason, the introduction of tropical tools often yields new insights into old familiar problems, leading either to counterexamples or to new methods and results; see for instance [115], [102]. In some applications, like optimal control, discrete event systems, or static analysis of programs, tropical objects do not appear through a limit procedure, but more directly as a modelling or computation/analysis tool; see for instance [112], [66], [89], [69].

Tropical methods are linked to the fields of positive systems and of metric geometry [105], [11]. Indeed, tropically linear maps are monotone (a.k.a. order-preserving). They are also nonexpansive in certain natural metrics (sup-norm, Hopf oscillation, Hilbert's projective metric, ...). In this way, tropical dynamical systems appear to be special cases of nonexpansive, positive, or monotone dynamical systems, which are studied as part of linear and non-linear Perron-Frobenius theory [95], [14]. Such dynamical systems are of fundamental importance in the study of repeated games [101]. Monotonicity properties are also essential in the understanding of the fixed points problems which determine program invariants by abstract interpretation [73]. The latter problems are actually somehow similar to the ones arising in the study of zero-sum games; see [7]. Moreover, positivity or monotonicity methods are useful in population dynamics, either in a discrete space setting [113] or in a PDE setting [67]. In such cases, solving tropical problems often leads to solutions or combinatorial insights on classical problems involving positivity conditions (e.g., finding equilibria of dynamical systems with nonnegative coordinates, understanding the qualitative and quantitative behavior of growth rates / Floquet eigenvalues [9], etc). Other applications of Perron-Frobenius theory originate from quantum information and control [107], [111].

3. Research Program

3.1. Optimal control and zero-sum games

The dynamic programming approach allows one to analyze one or two-player dynamic decision problems by means of operators, or partial differential equations (Hamilton–Jacobi or Isaacs PDEs), describing the time evolution of the value function, i.e., of the optimal reward of one player, thought of as a function of the initial state and of the horizon. We work especially with problems having long or infinite horizon, modelled by stopping problems, or ergodic problems in which one optimizes a mean payoff per time unit. The determination of optimal strategies reduces to solving nonlinear fixed point equations, which are obtained either directly from discrete models, or after a discretization of a PDE.

The geometry of solutions of optimal control and game problems Basic questions include, especially for stationary or ergodic problems, the understanding of existence and uniqueness conditions for the solutions of dynamic programming equations, for instance in terms of controllability or ergodicity properties, and more generally the understanding of the structure of the full set of solutions of stationary Hamilton–Jacobi PDEs and of the set of optimal strategies. These issues are already challenging in the one-player deterministic case, which is an application of choice of tropical methods, since the Lax-Oleinik semigroup, i.e., the evolution semigroup of the Hamilton-Jacobi PDE, is a linear operator in the tropical sense. Recent progress in the deterministic case has been made by combining dynamical systems and PDE techniques (weak KAM theory [79]), and also using metric geometry ideas (abstract boundaries can be used to represent the sets of solutions [90], [4]). The two player case is challenging, owing to the lack of compactness of the analogue of the Lax-Oleinik semigroup and to a richer geometry. The conditions of solvability of ergodic problems for games (for instance, solvability of ergodic Isaacs PDEs), and the representation of solutions are only understood in special cases, for instance in the finite state space case, through tropical geometry and non-linear Perron-Frobenius methods [54],[47], [14].

Algorithmic aspects: from combinatorial algorithms to the attenuation of the curse of dimensionality Our general goal is to push the limits of solvable models by means of fast algorithms adapted to large scale instances. Such instances arise from discrete problems, in which the state space may so large that it is only accessible through local oracles (for instance, in some web ranking applications, the number of states may be the number of web pages) [80]. They also arise from the discretization of PDEs, in which the number of states grows exponentially with the number of degrees of freedom, according to the "curse of dimensionality". A first line of research is the development of *new approximation methods for the value function*. So far, classical approximations by linear combinations have been used, as well as approximation by suprema of linear or quadratic forms, which have been introduced in the setting of dual dynamic programming and of the so called "max-plus basis methods" [81]. We believe that more concise or more accurate approximations may be obtained by unifying these methods. Also, some max-plus basis methods have been shown to *attenuate the curse of dimensionality* for very special problems (for instance involving switching) [98], [84]. This suggests that the complexity of control or games problems may be measured by more subtle quantities that the mere number of states, for instance, by some forms of metric entropy (for example, certain large scale problems have a low complexity owing to the presence of decomposition properties, "highway hierarchies", etc.). A second line of of our research is the development of *combinatorial algorithms*, to solve large scale zero-sum two-player problems with discrete state space. This is related to current open problems in algorithmic game theory. In particular, the existence of polynomial-time algorithms for games with ergodic payment is an open question. See e.g. [5] for a polynomial time average complexity result derived by tropical methods. The two lines of research are related, as the understanding of the geometry of solutions allows to develop better approximation or combinatorial algorithms.

3.2. Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

Several applications (including population dynamics [9] and discrete event systems [66], [71], [62]) lead to studying classes of dynamical systems with remarkable properties: preserving a cone, preserving an order, or being nonexpansive in a metric. These can be studied by techniques of non-linear Perron-Frobenius theory [14] or metric geometry [10]. Basic issues concern the existence and computation of the "escape rate" (which determines the throughput, the growth rate of the population), the characterizations of stationary regimes (non-linear fixed points), or the study of the dynamical properties (convergence to periodic orbits). Nonexpansive mappings also play a key role in the "operator approach" to zero-sum games, since the one-day operators of games are nonexpansive in several metrics, see [8].

3.3. Tropical algebra and convex geometry

The different applications mentioned in the other sections lead us to develop some basic research on tropical algebraic structures and in convex and discrete geometry, looking at objects or problems with a "piecewise-linear " structure. These include the geometry and algorithmics of tropical convex sets [64], [56], tropical semialgebraic sets [49], the study of semi-modules (analogues of vector spaces when the base field is replaced by a semi-field), the study of systems of equations linear in the tropical sense, investigating for instance the analogues of the notions of rank, the analogue of the eigenproblems [15], and more generally of systems of tropical polynomial equations. Our research also builds on, and concern, classical convex and discrete geometry methods.

3.4. Tropical methods applied to optimization, perturbation theory and matrix analysis

Tropical algebraic objects appear as a deformation of classical objects thought various asymptotic procedures. A familiar example is the rule of asymptotic calculus,

$$e^{-a/\epsilon} + e^{-b/\epsilon} \asymp e^{-\min(a,b)/\epsilon} \; , \qquad e^{-a/\epsilon} \times e^{-b/\epsilon} = e^{-(a+b)/\epsilon} \; , \tag{1}$$

when $\epsilon \to 0^+$. Deformations of this kind have been studied in different contexts: large deviations, zero-temperature limits, Maslov's "dequantization method" [97], non-archimedean valuations, log-limit sets and Viro's patchworking method [116], etc.

This entails a relation between classical algorithmic problems and tropical algorithmic problems, one may first solve the $\epsilon = 0$ case (non-archimedean problem), which is sometimes easier, and then use the information gotten in this way to solve the $\epsilon = 1$ (archimedean) case.

In particular, tropicalization establishes a connection between polynomial systems and piecewise affine systems that are somehow similar to the ones arising in game problems. It allows one to transfer results from the world of combinatorics to "classical" equations solving. We investigate the consequences of this correspondence on complexity and numerical issues. For instance, combinatorial problems can be solved in a robust way. Hence, situations in which the tropicalization is faithful lead to improved algorithms for classical problems. In particular, scalings for the polynomial eigenproblems based on tropical preprocessings have started to be used in matrix analysis [85], [88].

Moreover, the tropical approach has been recently applied to construct examples of linear programs in which the central path has an unexpectedly high total curvature [61], and it has also led to positive polynomial-time average case results concerning the complexity of mean payoff games. Similarly, we are studying semidefinite programming over non-archimedean fields [49], [29], with the goal to better understand complexity issues in classical semidefinite and semi-algebraic programming.

4. Application Domains

4.1. Discrete event systems (manufacturing systems, networks)

One important class of applications of max-plus algebra comes from discrete event dynamical systems [66]. In particular, modelling timed systems subject to synchronization and concurrency phenomena leads to studying dynamical systems that are non-smooth, but which have remarkable structural properties (nonexpansiveness in certain metrics , monotonicity) or combinatorial properties. Algebraic methods allow one to obtain analytical expressions for performance measures (throughput, waiting time, etc). A recent application, to emergency call centers, can be found in [62].

4.2. Optimal control and games

Optimal control and game theory have numerous well established applications fields: mathematical economy and finance, stock optimization, optimization of networks, decision making, etc. In most of these applications, one needs either to derive analytical or qualitative properties of solutions, or design exact or approximation algorithms adapted to large scale problems.

4.3. Operations Research

We develop, or have developed, several aspects of operations research, including the application of stochastic control to optimal pricing, optimal measurement in networks [109]. Applications of tropical methods arise in particular from discrete optimization [68], [70], scheduling problems with and-or constraints [103], or product mix auctions [114].

4.4. Computing program and dynamical systems invariants

A number of programs and systems verification questions, in which safety considerations are involved, reduce to computing invariant subsets of dynamical systems. This approach appears in various guises in computer science, for instance in static analysis of program by abstract interpretation, along the lines of P. and R. Cousot [73], but also in control (eg, computing safety regions by solving Isaacs PDEs). These invariant sets are often sought in some tractable effective class: ellipsoids, polyhedra, parametric classes of polyhedra with a controlled complexity (the so called "templates" introduced by Sankaranarayanan, Sipma and Manna [110]), shadows of sets represented by linear matrix inequalities, disjunctive constraints represented by tropical polyhedra [63], etc. The computation of invariants boils down to solving large scale fixed point problems. The latter are of the same nature as the ones encountered in the theory of zero-sum games, and so, the techniques developed in the previous research directions (especially methods of monotonicity, nonexpansiveness, discretization of PDEs, etc) apply to the present setting, see e.g. [83], [86] for the application of policy iteration type algorithms, or for the application for fixed point problems over the space of quadratic forms [7]. The problem of computation of invariants is indeed a key issue needing the methods of several fields: convex and nonconvex programming, semidefinite programming and symbolic computation (to handle semialgebraic invariants), nonlinear fixed point theory, approximation theory, tropical methods (to handle disjunctions), and formal proof (to certify numerical invariants or inequalities).

5. Highlights of the Year

5.1. Highlights of the Year

5.1.1. *Awards*

- The Gaspard Monge Programme for Optimization and Operations Research (PGMO), a corporate sponsorship of EDF operated by Fondation Mathématique Jacques Hadamard, coordinated by Stéphane Gaubert, received the "Grand Prix AEF – meilleures initiatives partagées Universités Entreprises".
- Mateusz Skomra received the Dodu prize (distinction for the best talk of a young researcher) at the conférence SMAI-MODE 2016.

6. New Software and Platforms

6.1. New Software

6.1.1. *Coq-Polyhedra*

Coq-Polyhedra is a library which aims at formalizing convex polyhedra in Coq. A description of the associated contributions can be found in Section 7.3.1. Coq-Polyhedra is distributed under the CeCILL-B licence, and can be found at https://github.com/nhojem/Coq-Polyhedra.

7. New Results

7.1. Optimal control and zero-sum games

7.1.1. *Fixed points of order preserving homogeneous maps and zero-sum games*
Participants: Marianne Akian, Stéphane Gaubert, Antoine Hochart.

The PhD work of Antoine Hochart [12] deals with the applications of methods of non-linear fixed point theory to zero-sum games.

A highlight of his PhD is the characterization of the property of ergodicity for zero-sum games. In the special "zero-player" case, i.e., for a Markov chain equipped with an additive functional (payment) of the trajectory, the ergodicity condition entails that the mean payoff is independent of the initial state, for any choice of the payment. In the case of finite Markov chains, ergodicity admits several characterizations, including a combinatorial one (the uniqueness of the final class). This carries over to the two player case: ergodicity is now characterized by the absence of certain pairs of conjugate invariant sets (dominions), and it can be checked using directed hypergraphs algorithms. This leads to an explicit combinatorial sufficient condition for the solvability of the "ergodic equation", which is the main tool in the numerical approach of the mean payoff problem. These results appeared in [59], [58], [60]. A more general approach was developed in [30], in which zero-sum games are now studied abstractly in terms of accretive operators. This allows one to show that the bias vector (the solution of the ergodic equation) is unique for a generic perturbation of the payments.

Another series of results of the thesis concern the finite action space, showing that the set of payments for which the bias vector is not unique coincides with the union of lower dimensional cells of a polyhedral complex, which an application to perturbation schemes in policy iteration [47].

A last result of the thesis is a representation theorem for "payment free" Shapley operators, showing that these are characterized by monotonicity and homogeneity axioms [48]. This extends to the two-player case known representation theorems for risk measures.

7.1.2. *Probabilistic and max-plus approximation of Hamilton-Jacobi-Bellman equations*
Participants: Marianne Akian, Eric Fodjo.

The PhD thesis of Eric Fodjo concerns stochastic control problems obtained in particular in the modelisation of portfolio selection with transaction costs. The dynamic programming method leads to a Hamilton-Jacobi-Bellman partial differential equation, on a space with a dimension at least equal to the number of risky assets. The curse of dimensionality does not allow one to solve numerically these equations for a large dimension (greater to 5). We propose to tackle these problems with numerical methods combining policy iterations, probabilistic discretisations, max-plus discretisations, in order to increase the possible dimension. Another solution is to replace policy iterations by an approximation with optimal switching problems.

In [27], [26] (also presented in [35], [23]), we consider fully nonlinear Hamilton-Jacobi-Bellman equations associated to diffusion control problems with finite horizon involving a finite set-valued (or switching) control and possibly a continuum-valued control. We construct a lower complexity probabilistic numerical algorithm by combining the idempotent expansion properties obtained by McEneaney, Kaise and Han [93], [99] for solving such problems with a numerical probabilistic method such as the one proposed by Fahim, Touzi and Warin [78] for solving some fully nonlinear parabolic partial differential equations, when the volatility does not oscillate too much. Numerical tests on a small example of pricing and hedging an option are presented. Moreover, more recently, we improved the method of Fahim, Touzi and Warin to allow one to solve fully nonlinear parabolic partial differential equations with general volatilities.

7.2. Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

7.2.1. *Isometries of the Hilbert geometry*
Participant: Cormac Walsh.

In a collaboration with Bas Lemmens (Kent University, UK), we have been studying the Hilbert geometry in finite dimensions, especially its horofunction boundary and isometry group. The book chapter [117] contains a survey of this work. However, the infinite dimensional case is also interesting, and has been used as a tool for many years in non-linear analysis. Despite this, very little is known about the geometry of these spaces when the dimension is infinite.

An example of a problem in which we are interested is the following. In finite dimension it is known that a Hilbert geometry is isometric to a normed space if and only if it is a simplex. We have shown [118] that, more generally, a Hilbert geometry is isometric to a Banach space if and only if it is the cross-section of a positive cone, that is, the cone of positive continuous functions on some compact topological space. To solve this problem we found it useful to study the horofunction boundary in the infinite-dimensional case.

We are continuing to study similar problems in relation to this topic in collaboration with Bas Lemmens of the University of Kent.

7.2.2. *Volume growth in the Hilbert geometry*
Participant: Cormac Walsh.

In a collaboration with Constantin Vernicos of Université Montpellier 2, we are investigating how the volume of a ball in a Hilbert geometry grows as its radius increases. Specifically, we are studying the volume entropy

$$\lim_{r \to \infty} \frac{\log \operatorname{Vol} B(x, r)}{r}, \tag{2}$$

where $B(x, r)$ is the ball with center x and radius r, and Vol denotes some notion of volume, for example, the Holmes–Thompson or Busemann definitions. Note that the entropy does not depend on the particular choice of x, nor on the choice of the volume. It is known that the hyperbolic space, or indeed any Hilbert geometry with a C^2-smooth boundary of strictly positive curvature, has entropy $n-1$, where n is the dimension, and it has recently been proved that this is the maximal entropy possible for Hilbert geometries of the given dimension.

Constantin Vernicos has shown that, in dimension 2 and 3, the volume entropy of a Hilbert geometry on a convex body is equal to exactly twice the *approximability* of the body, that is, the power of $1/\epsilon$ governing the growth of the number of vertices needed to approximate the body by a polytope within ϵ, as ϵ decreases.

Studying polytopal Hilbert geometries, we have demonstrated [53] a close relation between the volume and the number of *flags* of the polytope, more precisely, that the volume of large balls is asymptotically proportional to the number of flags. This suggested to us defining a new notion of approximability using flags rather than vertices. We have shown [53] that the volume entropy of a Hilbert geometry on a convex body is equal to exactly twice this *flag-approximability* in all dimensions. This implies in particular that the volume entropy of a convex body is equal to that of its dual.

7.2.3. *The set of minimal upper bounds of two matrices in the Loewner order*
Participant: Nikolas Stott.

A classical theorem of Kadison shows that the space of symmetric matrices equipped with the Loewner order is an anti-lattice, meaning that two matrices have a least upper bound if and only if they are comparable. In [52], we refined this theorem by characterizing the set of minimal upper bounds: we showed that it is homeomorphic to the quotient space $O(p) \smallsetminus O(p,q)/O(q)$, where $O(p,q)$ denotes the orthogonal group associated to the quadratic form with signature (p,q), and $O(p)$ denotes the standard pth orthogonal group.

7.2.4. *Checking the strict positivity of Kraus maps is NP-hard*
Participant: Stéphane Gaubert.

In collaboration with Zheng Qu (now with HKU, Hong Kong), I studied several decision problems arising from the spectral theory of Kraus maps (trace preserving completely positive maps), acting on the cone of positive semidefinite matrices. The latter appear in quantum information. We showed that checking the irreducibility (absence of non-trivial invariant face of the cone) and primitivity properties (requiring the iterates of the map to send the cone to its interior) can be checked in polynomial time, whereas checking positivity (whether the map sends the cone to its interior) is NP-hard. In [17], we studied complexity issues related to Kraus maps, and showed in particular that checking whether a Kraus map sends the cone to its interior is NP-hard.

7.3. Tropical algebra and convex geometry

7.3.1. *Formalizing convex polyhedra in Coq*
Participants: Xavier Allamigeon, Ricardo Katz [Conicet, Argentine].

We formalize a certain fragment of the theory of convex polyhedra and their combinatorial properties. Our motivation is that convex polyhedra are involved in a wide range of analysis techniques such as in formal verification, and that their combinatorial properties are used to establish more fundamental results, especially in tropical geometry.

This formalization has been conducted in Coq using the Mathematical Components library. We have implemented a full formalization of the simplex algorithm, which allows to make several key properties of convex polyhedra (feasibility, unboundedness, etc) decidable. From this, we have deduced a formal proof of strong duality theorem in linear programming, and of Farkas lemma. We also have a formal implementation of Motzkin's double description method, which provides a constructive way to prove Minkowski theorem for polyhedra.

7.3.2. *Tropical totally positive matrices*
Participants: Stéphane Gaubert, Adi Niv.

In [50], we investigate the tropical analogues of totally positive and totally non-negative matrices, i.e, the images by the valuation of the corresponding classes of matrices over a non-archimedean field. We show that tropical totally positive matrices essentially coincide with the Monge matrices (defined by the positivity of 2×2 tropical minors), arising in optimal transport. These results have been presented in [41], [40].

7.3.3. *Tropical compound matrix identities*

Participants: Marianne Akian, Stéphane Gaubert, Adi Niv.

In [55], [57], we proved some identities on matrices using a weak and a strong transfer principles. In the present work, we prove identities on compound matrices in extended tropical semirings. Such identities include analogues to properties of conjugate matrices, powers of matrices and $\mathrm{adj}\,(A)\det\,(A)^{-1}$, all of which have implications on the eigenvalues of the corresponding matrices. A tropical Sylvester-Franke identity is provided as well. Even though part of these identities hold over any commutative ring, they cannot be adjusted to semirings with symmetry using the existing weak and strong transfer principles. Here, we provide the proofs by means of graph theory arguments.

7.3.4. *Supertropical algebra*

Participant: Adi Niv.

Several properties of matrices over the tropical algebra are studied using the supertropical algebra introduced in [92].

The only invertible matrices in tropical algebra are diagonal matrices, permutation matrices and their products. However, the pseudo-inverse A^∇, defined as $\frac{1}{\det(A)}\,\mathrm{adj}\,(A)$, with $\det(A)$ being the tropical permanent, inherits some classical algebraic properties and has some surprising new ones. In [104], defining B and B' to be tropically similar if $B' = A^\nabla BA$, we examine the characteristic (max-)polynomials of tropically similar matrices as well as those of pseudo-inverses. Other miscellaneous results include a new proof of the identity for $\det(AB)$ and a connection to stabilization of the powers of definite matrices.

In a joint work with Louis Rowen (Bar Ilan Univ.) [21], we study the pathology that causes tropical eigenspaces of distinct supertropical eigenvalues of a non-singular matrix A, to be dependent. We show that in lower dimensions the eigenvectors of distinct eigenvalues are independent, as desired. The index set that differentiates between subsequent essential monomials of the characteristic polynomial, yields an eigenvalue λ, and corresponds to the columns of the eigenmatrix $A + \lambda I$ from which the eigenvectors are taken. We ascertain the cause for failure in higher dimensions, and prove that independence of the eigenvectors is recovered in case the "difference criterion" holds, defined in terms of disjoint differences between index sets of subsequent coefficients. We conclude by considering the eigenvectors of the matrix $A^\nabla := \frac{1}{\det(A)}\mathrm{adj}(A)$ and the connection of the independence question to generalized eigenvectors.

7.3.5. *Volume and integer points of tropical polytopes*

Participants: Marie Maccaig, Stéphane Gaubert.

We investigated the volume of tropical polytopes, as well as the number of integer points contained in integer polytopes. We proved that even approximating these values for a tropical polytope given by its vertices is hard, with no approximation algorithm with factor $2^{\mathrm{poly}(m,n)}$ existing. We further proved the $\sharp$P-hardness for the analogous problems for tropical polytopes instead defined by inequalities. We also investigated the relation between the set of integer points of a tropical polytope and the image by the valuation of polytopes over the nonarchimedean field of Puiseux series.

7.3.6. *Primal dual pair of max-algebraic integer linear programs (MLP)*

Participant: Marie Maccaig.

There are known weak and strong duality theorems for max-algebraic linear programs. I investigated the integer versions of these problems; considering the impact of requiring integer solutions instead of real solutions. I proved a tight bound on the duality gap for a pair of integer solutions to the primal and dual MLPs, and searched for conditions on when the optimal values of the integer primal and dual MLPs coincide.

7.3.7. *Tropical Jacobi identity*

Participants: Marie Maccaig, Adi Niv.

In a joint work with Sergei Sergeev (Birmingham), we investigated the combinatorial interpretation for the Tropical Jacobi identity. Inspired by Butkovic's paper, "Max-algebra, the algebra of combinatorics?" and many other links between max-algebra and combinatorics, we try to link this tropical quantity to a new type of multiple assignment problem.

7.4. Tropical methods applied to optimization, perturbation theory and matrix analysis

7.4.1. *Majorization inequalities for valuations of eigenvalues using tropical algebra*
Participants: Marianne Akian, Stéphane Gaubert.

We consider a matrix with entries over the field of Puiseux series, equipped with its non-archimedean valuation (the leading exponent). In [13], with Ravindra Bapat (Univ. New Delhi), we establish majorization inequalities relating the sequence of the valuations of the eigenvalues of a matrix with the tropical eigenvalues of its valuation matrix (the latter is obtained by taking the valuation entrywise). We also show that, generically in the leading coefficients of the Puiseux series, the precise asymptotics of eigenvalues, eigenvectors and condition numbers can be determined. For this, we apply diagonal scalings constructed from the dual variables of a parametric optimal assignment constructed from the valuation matrix.

In recent works with Andrea Marchesini and Françoise Tisseur (Manchester University), we use the same technique to establish an archimedean analogue of the above inequalities, which applies to matrix polynomials with coefficients in the field of complex numbers, equipped with the modulus as its valuation. This allows us in particular to improve the accuracy of the numerical computation of the eigenvalues of such matrix polynomials.

In [15], with Meisam Sharify (IPM, Tehran, Iran), we also establish log-majorization inequalities of the eigenvalues of matrix polynomials using the tropical roots of some scalar polynomials depending only on the norms of the matrix coefficients. This extends to the case of matrix polynomials some bounds obtained by Hadamard, Ostrowski and Pólya for the roots of scalar polynomials.

These works have been presented in [22].

7.4.2. *Tropicalization of the central path and application to the complexity of interior point methods*
Participants: Xavier Allamigeon, Stéphane Gaubert.

This work is in collaboration with Pascal Benchimol (now with EDF Labs) and Michael Joswig (TU-Berlin).

In optimization, path-following interior point methods are driven to an optimal solution along a trajectory called the central path. The *central path* of a linear program $\mathrm{LP}(A, b, c) \equiv \min\{c \cdot x \mid Ax \leq b, \ x \geq 0\}$ is defined as the set of the optimal solutions (x^{μ}, w^{μ}) of the barrier problems:

$$\text{minimize} \quad c \cdot x - \mu\left(\sum_{j=1}^{n} \log x_j + \sum_{i=1}^{m} \log w_i\right)$$
$$\text{subject to} \quad Ax + w = b, \ x > 0, \ w > 0$$

While the complexity of interior point methods is known to be polynomial, an important question is to study the number of iterations which are performed by interior point methods, in particular whether it can be bounded by a polynomial in the dimension (mn) of the problem. This is motivated by one of Smale's problems, on the existence of a strongly polynomial complexity algorithm for linear programming. So far, this question has been essentially addressed though the study of the curvature of the central path, which measures how far a path differs from a straight line, see [75], [74], [77], [76]. In particular, by analogy with the classical Hirsch conjecture, Deza, Terlaky and Zinchenko [76] conjectured that $O(m)$ is also an upper bound for the total curvature.

In a work of X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig, we study the tropicalization of the central path. The *tropical central path* is defined as the logarithmic limit of the central paths of a parametric family of linear programs $\mathrm{LP}(A(t), b(t), c(t))$, where the entries $A_{ij}(t)$, $b_i(t)$ and $c_j(t)$ are definable functions in an o-minimal structure called the *Hardy field*.

A first contribution is to provide a purely geometric characterization of the tropical central path. We have shown that the tropical analytic center is the greatest element of the tropical feasible set. Moreover, any point of the tropical central path is the greatest element of the tropical feasible set intersected with a sublevel set of the tropical objective function.

Thanks to this characterization, we identify a class of path-following interior-point methods which are not strongly polynomial. This class corresponds to primal-dual interior-point methods which iterates in the so-called "wide" neighborhood of the central path arising from the logarithmic barrier. It includes short step, long step as well as predictor-corrector types of interior-point methods. In more details, we establish a lower bound on the number of iterations of these methods, expressed in terms of the number of tropical segments constituting the tropical central path. In this way, we exhibit a family of linear programs with $3d + 1$ inequalities in dimension $2d$ on which the aforementioned interior point methods require $\Omega(2^d)$ iterations. The same family provides a counterexample to Deza, Terlaky and Zinchenko's conjecture, having a total curvature in $\Omega(2^d)$.

A first part of these results is in the preprint [61], further results been presented in [32].

7.4.3. *Tropical approach to semidefinite programming*

Participants: Xavier Allamigeon, Stéphane Gaubert, Mateusz Skomra.

Semidefinite programming consists in optimizing a linear function over a spectrahedron. The latter is a subset of $\mathbb{R}^n$ defined by linear matrix inequalities, i.e., a set of the form

$$\left\{ x \in \mathbb{R}^n : Q^{(0)} + x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \succeq 0 \right\}$$

where the $Q^{(k)}$ are symmetric matrices of order m, and $\succeq$ denotes the Loewner order on the space of symmetric matrices. By definition, $X \succeq Y$ if and only if $X - Y$ is positive semidefinite.

Semidefinite programming is a fundamental tool in convex optimization. It is used to solve various applications from engineering sciences, and also to obtain approximate solutions or bounds for hard problems arising in combinatorial optimization and semialgebraic optimization.

A general issue in computational optimization is to develop combinatorial algorithms for semidefinite programming. Indeed, semidefinite programs are usually solved via interior point methods. However, the latter provide an approximate solution in a polynomial number of iterations, provided that a strictly feasible initial solution. Semidefinite programming becomes a much harder matter if one requires an exact solution. The feasibility problem belongs to $\mathsf{NP}_{\mathbb{R}} \cap \mathsf{coNP}_{\mathbb{R}}$, where the subscript $\mathbb{R}$ refers to the BSS model of computation. It is not known to be in NP in the bit model.

We address semidefinite programming in the case where the field $\mathbb{R}$ is replaced by a nonarchimedean field, like the field of Puiseux series. In this case, methods from tropical geometry can be applied and are expected to allow one, in generic situations, to reduce semialgebraic problems to combinatorial problems, involving only the nonarchimedean valuations (leading exponents) of the coefficients of the input.

To this purpose, we first study tropical spectrahedra, which are defined as the images by the valuation of nonarchimedean spectrahedra. We establish that they are closed semilinear sets, and that, under a genericity condition, they are described by explicit inequalities expressing the nonnegativity of tropical minors of order 1 and 2. These results are gathered in the preprint [49].

Then, we show that the feasibility problem for a generic tropical spectrahedron is equivalent to solving a stochastic mean payoff game (with perfect information). The complexity of these games is a long-standing open problem. They are not known to be polynomial, however they belong to the class NP ∩ coNP, and they can be solved efficiently in practice. This allows to apply stochastic game algorithms to solve nonarchimedean semidefinite feasibility problems. We obtain in this way both theoretical bounds and a practicable method which solves some large scale instances. Part of this latter work has been published in the proceedings of the conference ISSAC 2016 [29].

7.5. Applications

7.5.1. *Geometry of the Loewner order and application to the synthesis of quadratic invariants in static analysis of program*

Participants: Xavier Allamigeon, Stéphane Gaubert, Nikolas Stott.

This work is in collaboration with Éric Goubault and Sylvie Putot (from LIX).

We introduce a new numerical abstract domain based on ellipsoids designed for the formal verification of switched linear systems. The novelty of this domain does not consist in the use of ellipsoids as abstractions, but rather in the fact that we overcome two key difficulties which so far have limited the use of ellipsoids in abstract interpretation. The first issue is that the ordered set of ellipsoids does not constitute a lattice. This implies that there is a priori no canonical choice of the abstraction of the union of two sets, making the analysis less predictable as it relies on the selection of good upper bounds. The second issue is that most recent works using on ellipsoids rely on LMI methods. The latter are efficient on moderate size examples but they are inherently limited by the complexity of interior point algorithms, which, in the case of matrix inequality problems, do not scale as well as for linear programming or second order cone programming problems.

We developed a new approach, in which we reduce the computation of an invariant to the determination of a fixed point, or eigenvector, of a non-linear map that provides a safe upper-approximation of the action induced by the program on the space of quadratic forms. This allows one to obtain invariants of systems of sized inaccessible by LMI methods, at the price of a limited loss of precision. A key ingredient here is the fast computation of least upper bounds in Löwner ordering, by an algebraic algorithm. This relies on the study of the geometry of the space of quadratic forms (Section 7.2.3).

A first part of this work is described in the article [16], which is the extended version of [65] which won the best paper award at the conference EMSOFT 2015. Followup work is dealing with the extension of these results to switched affine systems with guards.

7.5.2. *Performance evaluation of an emergency call center based on tropical polynomial systems*

Participants: Xavier Allamigeon, Vianney Boeuf, Stéphane Gaubert.

This work arose from a question raised by Régis Reboul from Préfecture de Police de Paris (PP), regarding the analysis of the projected evolution of the treatment of emergency calls (17-18-112). This work benefited from the help of LtL Stéphane Raclot, from Brigade de Sapeurs de Pompiers de Paris (BSPP). It is part of the PhD work of Vianney Bœuf, carried out in collaboration with BSPP.

We introduced an algebraic approach which allows to analyze the performance of systems involving priorities and modeled by timed Petri nets. Our results apply to the class of Petri nets in which the places can be partitioned in two categories: the routing in certain places is subject to priority rules, whereas the routing at the other places is free choice.

In [62], we introduced a discrete model, showing that the counter variables, which determine the number of firings of the different transitions as a function of time, are the solutions of a piecewise linear dynamical system. Moreover, we establish that in the fluid approximation of this model, the stationary regimes are precisely the solutions of a set of lexicographic piecewise linear equations, which constitutes a polynomial system over a tropical (min-plus) semifield of germs.

In essence, this result shows that computing stationary regimes reduces to solving tropical polynomial systems. Solving tropical polynomial systems is one of the most basic problems of tropical geometry. The latter provides insights on the nature of solutions, as well as algorithmic tools. In particular, the tropical approach allows one to determine the different congestion phases of the system.

We applied this approach to a case study relative to the project led by Préfecture de Police de Paris, involving BSPP, of a new organization to handle emergency calls to Police (number 17), Firemen (number 18), and untyped emergency calls (number 112), in the Paris area. We initially introduced, in [62], a simplified model of emergency call center, and we concentrated on the analysis of an essential feature of the organization: the two level emergency procedure. Operators at level 1 initially receive the calls, qualify their urgency, handle the non urgent ones, and transfer the urgent cases to specialized level 2 operators who complete the instruction. We solved the associated system of tropical polynomial equations and arrived at an explicit computation of the different congestion phases, depending on the ratio of the numbers of operators of level 2 and 1.

We subsequently developed a more complex model, taking into account the different characteristics of the calls to 17 and 18, and developed a realistic simulation tool to validate the results. Moreover, in [28], we developed an alternative model, relying on fluid Petri nets (dynamical systems with piecewise affine vector fields). We showed that the fluid and discrete models have the same stationary regimes, and that some pathological features of the discrete model (anomalous periodic orbits appearing under certain arithmetical conditions) vanish in the fluid Petri net case.

7.5.3. *Smart Data Pricing*

Participants: Marianne Akian, Jean-Bernard Eytard.

This work is in collaboration with Mustapha Bouhtou (Orange Labs).

The PhD work of Jean-Bernard Eytard concerns the optimal pricing of data trafic in mobile networks. We developed a bilevel programming approach, allowing to an operator to balance the load in the network through price incentives. We showed that a subclass of bilevel programs can be solved in polynomial time, by combining methods of tropical geometry and of discrete convexity. This work has been presented in [31].

8. Bilateral Contracts and Grants with Industry

8.1. Bilateral Contracts with Industry

- Yield management methods applied to the pricing of data trafic in mobile networks. CRE (research contract) with Orange Labs (Orange Labs partner: Mustapha Bouhtou).
- Decentralized mechanisms of operation of power systems: equilibria and efficiency. A collaboration started on this topic at the fall, Nadia Oujdane, Olivier Beaude, and Riadh Zorgati from EDF-labs. This leads to the PhD work of Paulin Jacquot, supervised by Stéphane Gaubert (starting CIFRE PhD).

9. Partnerships and Cooperations

9.1. National Initiatives

9.1.1. *ANR*

- Participation of Cormac Walsh to the ANR white project FINSLER (Géométrie de Finsler et applications), 2012-2016.
- Projet ANR CAFEIN (Combinaison d'approches formelles pour l'étude d'invariants numériques), responsable P.L. Garoche. Partenaires : ONERA, CEA LIST, ENSTA Paristech, Inria Saclay (Max-plus, Toccata, Parkas), Université de Perpignan, Prover, Rockwell Collins France.

- Projet ANR MALTHY (Méthodes ALgébriques pour la vérification de modèles Temporisés et HYbrides), responsable T. Dang. Partenaires : Verimag, CEA LIST, Inria Rennes, Inria Saclay, VISEO/Object Direct.
- Projet ANR DEMOCRITE ("DEmonstrateur d'un MOteur de Couverture des Risques sur un TErritoire), responsable Emmanuel Lapébie (CEA). Partenaires : CEA-GRAMAT, BSPP, Inria Saclay (Maxplus), Institut PPRIME - UPR3346 (CNRS, Univ. Poitiers, ISAE-ENSMA), IPSIS, SYSTEL, ARMINES-E.M. Alès-ISR, CERDACC (Univ. de Haute-Alsace).

9.1.2. *Programme Gaspard Monge pour l'Optimisation*

- Projet intitulé "Méthodes tropicales pour l'optimisation", responsable X. Allamigeon, faisant intervenir M. Akian, V. Boeuf, S. Gaubert, A.Hochart, R. Katz, et M. Skomra.

9.1.3. *iCODE (Institut pour le Contrôle et la Décision de l'Idex Paris-Saclay)*

- White project "Stabilité et stabilisation des systèmes commutés" (Oct 2014-June 2016), including M. Akian, X. Allamigeon, S. Gaubert, and members of EPI Geco, L2S, LIX, LSV (ENS Cachan), UVSQ.

9.2. International Research Visitors

9.2.1. *Visits of International Scientists*

- Ricardo Katz (Conicet and Cifasis, Argentina), May–June 2016
- Rajendra Bhatia (Indian Statistical Institute, New Delhi), 2 weeks in June 2016.
- Vladimir Gurvich (Rutgers), 2 weeks in Dec 2016.

9.2.2. *Visits to International Teams*

9.2.2.1. Research Stays Abroad

- S. Gaubert, invitation of one week to HKU, Hong-Kong, collaboration with Zheng Qu.

10. Dissemination

10.1. Promoting Scientific Activities

10.1.1. *Scientific Events Organisation*

10.1.1.1. General Chair, Scientific Chair

- S. Gaubert co-organized, jointly with D. Grigoriev (CNRS, Lille), M. Joswig (TU Berlin) and T. Theobald (Frankfurt), a Dagstuhl workshop, on "Effectivity in tropical mathematics, and beyond".

10.1.1.2. Member of the Organizing Committees

- M. Akian co-organized a workshop on hybrid systems, IHP, 2016.
- S. Gaubert co-organizes the "Séminaire Parisien d'Optimisation".
- S. Gaubert, co-organized with S. Charousset (EDF) the PGMO days at EDF labs Paris-Saclay.
- X. Allamigeon co-organized two invited sessions on semidefinite programming and tropical methods at the conference PGMO Days.

10.1.2. *Scientific Events Selection*

10.1.2.1. Chair of Conference Program Committees

- S. Gaubert, president of the scientific committee of SMAI-MODE 2016 (Toulouse, March 2016).

10.1.3. Journal

10.1.3.1. Member of the Editorial Boards

- S. Gaubert is member of the editorial committee of the collection Mathématiques et Applications, SMAI and Springer.
- S. Gaubert is associate editor of Linear and Multilinear Algebra.
- S. Gaubert is associate editor of RAIRO Operations research.

10.1.4. Invited Talks

- M. Akian, Keynote lecture at ETAMM 2016.
- S. Gaubert, invited lecture at the 2016 Conference on Applied Mathematics, Hong Kong University.

10.1.5. Leadership within the Scientific Community

- S. Gaubert coordinates the Gaspard Monge Programme for Optimization and Operations Research (PGMO), a corporate sponsorship of EDF operated by Fondation Mathématique Jacques Hadamard at Paris-Saclay. The goal of the program is to help to develop the research community in these fields, connecting academic and industrial researchers. It includes a research initiative on energy, led by S. Charousset from EDF (IROE, funding focused projects on the optimization of energy), and a subprogram dedicated to basic research (PRMO, funding smaller size projects). Projects are selected after an open call, instructed by the scientific commitee of PGMO. The program organizes advanced invited lectures for PhD students and researchers (in 2016, lectures by Yuri Nesterov from Louvain and Jean-Bernard Lasserre from LAAS), a regular seminar, and an annual conference (PGMO days, 250 participants in 2016). The program is currently being renewed, with an opening to new industrial partners interested by optimization. See https://www.fondation-hadamard.fr/PGMO for more information on PGMO.

10.1.6. Research Administration

- M. Akian :
  - Member of the "comité de liaison SMAI-MODE" since June 2015.
- S. Gaubert :
  - Coordinator of PGMO (Gaspard Monge Program for Optimization and Operations Research, a corporate sponsorship of EDF operated by FMJH).
  - Member of the scientific council of CMAP.
- X. Allamigeon:
  - Member of the scientific committee of Inria Saclay – Ile-de-France.
  - Member of the laboratory council of CMAP.

10.2. Teaching - Supervision - Juries

10.2.1. Teaching

- M. Akian
  - Course "Markov decision processes: dynamic programming and applications" joint between (3rd year of) ENSTA and M2 "Mathématiques et Applications", U. Paris Saclay, "Optimization", and shared with Jean-Philippe Chancelier (ENPC), 15 hours each.
- X. Allamigeon
  - Petites classes et encadrement d'enseignements d'approfondissement de Recherche Opérationnelle en troisième année à l'École Polytechnique (programme d'approfondissement de Mathématiques Appliquées) (niveau M1).

- Cours du M2 "Optimisation" de l'Université Paris Saclay, cours partagé avec Céline Gicquel et Dominique Quadri (LRI, Université Paris Sud).
- Co-responsabilité du programme d'approfondissement en mathématiques appliquées (troisième année) à l'École Polytechnique.

- V. Boeuf
  - Petite classe du cours de tronc commun de 1ere année "Introduction à l'optimisation" de l'École des ponts (ENPC), niveau L3.

- S. Gaubert
  - Course "Systèmes à Événements Discrets", option MAREVA, ENSMP.
  - Course "Algèbre max-plus pour le contrôle optimal et les jeux" of "Parcours Optimisation, Jeux et Dynamique" (ODJ) of M2 "Mathématiques et Applications" of Paris 6 University and École Polytechnique.
  - Lecture of Operations Research, third year of École Polytechnique. The lectures notes were published this year as a book [46].

- A. Hochart
  - Cours de niveau L1 et L2 à l'Univ. Paris Diderot (Paris VII), dans le cadre d'un monitorat (34h).

- M. Skomra
  - Petite classe du cours de tronc commun de 1ere année "Introduction à l'optimisation" de l'École des ponts (ENPC), niveau L1.

- N. Stott
  - Cours et TD "Introduction à la programmation graphique en C++", option MAREVA et semaine Athens à l'ENSMP (École des Mines de Paris), niveau M1.

10.2.2. *Supervision*

- PhD: Antoine Hochart, registered at École Polytechnique, since October 2013, thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian, defended on 14 Nov 2016.
- PhD in progress : Eric Fodjo, registered at École Polytechnique, since October 2013, thesis supervisor: Marianne Akian.
- PhD in progress : Nikolas Stott, registered at École Polytechnique, since October 2014, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon, Éric Goubault, Sylvie Putot.
- PhD in progress : Vianney Boeuf, registered at École Polytechnique, since October 2014, thesis supervisor: Stéphane Gaubert, cosupervision: Stéphane Raclot (BSPP), Marianne Akian, Xavier Allamigeon.
- PhD in progress : Mateusz Skomra, registered at Univ. Paris Saclay since October 2015, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.
- PhD in progress : Jean-Bernard Eytard, registered at Univ. Paris Saclay since October 2015, thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian, Mustapha Bouhtou.
- PhD in progress: Paulin Jacquot, registered at Univ. Paris Saclay since November 2016, thesis supervisor: Stéphane Gaubert, cosupervision: Nadia Oujdane, Olivier Beaude (EDF).

10.2.3. *Juries*

- M. Akian
  - Vice-president of the jury of the 2016 competition for CR2 positions of Inria Saclay–Île-de-France.
  - Member of the jury selecting the 2016 PGMO PhD price.
  - Jury of the PhD thesis of A. Hochart (X, Palaiseau, November 2016).

- S. Gaubert
  - Member of hiring commitee (Professor position) at Paris 6 University.
  - Member of hiring commitee (Assistant Professor position) at Limoges University.
  - Jury of the HdR of A. Auger (Saclay, 2016).
  - Jury of the HdR of B. Scherrer (Nancy, reviewer, 2016).
  - Jury of the PhD thesis of C. Josz (Paris 6, reviewer, 2016).
  - Jury of the PhD thesis of A. Parmentier (ENPC, reviewer, 2016).
  - Jury of the PhD thesis of A. Hochart (X, Palaiseau, 2016).

10.3. Popularization

- J.P. Quadrat :
  - Webmaster of the site http://www.maxplus.org, dedicated to max-plus algebra.

10.4. Conferences, Seminars

- M. Akian
  - Conference "Emerging Trends in Applied Mathematics and Mechanics" (ETAMM), Perpignan, May 30 - June 3, 2016, Title of the talk: "Majorization inequalities for valuations of eigenvalues using tropical algebra".
  - MTNS 2016 (22nd International Symposium on Mathematical Theory of Networks and Systems), Minneapolis, July 12-15, 2016, Title of the talk: "Solving Hamilton-Jacobi-Bellman equations by combining a max-plus linear approximation and a probabilistic numerical method".
  - Workshop "Numerical methods for Hamilton-Jacobi equations in optimal control and related fields" at the Radon Institute, Austrian Academy of Sciences, Linz, Austria, Nov. 21 - Nov. 25, 2016. Title of the talk: "Solving Hamilton-Jacobi-Bellman equations by combining a max-plus linear approximation and a probabilistic numerical method".
  - Dagstuhl Seminar "Algorithms and Effectivity in Tropical Mathematics and Beyond", Nov. 28 - Dec. 02, 2016. Title of the talk: "Majorization inequalities for valuations of eigenvalues using tropical algebra".
- X. Allamigeon
  - Journée Equipe McTAO, Inria Sophia Antipolis, January 18, 2016. Title of the talk: "Long and winding central paths".
  - Conference "Emerging Trends in Applied Mathematics and Mechanics" (ETAMM), Perpignan, May 30 - June 3, 2016. Title of the talk: "Long and winding central paths".
  - Groupe de travail combinatoire du Plateau de Saclay, June 8, 2016. Title of the talk: "Long and winding central paths".
  - Dagstuhl Seminar "Algorithms and Effectivity in Tropical Mathematics and Beyond", Nov. 28 - Dec. 02, 2016. Title of the talk: "Log-barrier interior-point methods are not strongly polynomial".
  - Séminaire Parisien d'Optimisation, Paris, December 12, 2016. Title of the talk: "Log-barrier interior-point methods are not strongly polynomial".
- V. Boeuf
  - Congrès annuel de la société Française de Recherche Opérationnelle et d'Aide à la Décision (ROADEF), Compiègne, February 10-12, 2016. Title of the talk: "Évaluation de performance en réception d'appels d'urgence : débits asymptotiques dans un réseau de Pétri avec priorités.".

–   Conference on Performance Evaluation Methodologies and Tools (VALUETOOLS), Taormina, October 25-28, 2016. Title of the talk: "Stationary solutions of discrete and continuous Petri nets with priorities".

- J.B. Eytard
    –   PGMO Days, Nov. 8-9, 2016, Palaiseau. Title of the talk: "Price incentives in mobile networks: a tropical approach".

- E. Fodjo
    –   9th European Summer School in Financial Mathematics, Aug. 29- Sep. 2, 2016, St Petersbourg, Russia. Title of the talk: "A probabilistic max-plus numerical method for solving stochastic control problems".
    –   SIAM Conference on Financial Mathematics & Engineering, Nov. 17-19, 2016, Austin, Texas, USA. Title of the talk: "A probabilistic max-plus numerical method for solving stochastic control problems".
    –   55th Conference on Decision and Control (CDC 2016), Dec. 12-14, 2016, Las Vegas, USA. Title of the talk: "A probabilistic max-plus numerical method for solving stochastic control problems".

- S. Gaubert
    –   International Conference on Tensors, Matrices and their Applications, Chern Institute of Mathematics, Nankai University, Tianjin, 20-24 May 2016. Title of the talk: "Ergodicity conditions in non-linear Perron-Frobenius theory and application to nonnegative tensors".
    –   SIAM Conference on Discrete Mathematics, Atlanta, June 6-10, 2016. Title of the talk: "Stochastic mean payoff games are tropical semidefinite programs".
    –   Workshop: Advances in Convex Analysis and Optimization Erice July 6-11, 2016. Title of the talk: "Zero-sum games, non-archimedean convexity and sinuous central paths".
    –   2016 Conference on Applied Mathematics, The University of Hong Kong Aug 23-26, 2016. Title of the talk: "Zero-sum games, non-archimedean convexity and sinuous central paths".
    –   Seminar of the algebraic geometry group at the University of Hong Kong, Aug 24, 2016. Title of the talk. "Tropical spectrahedra".
    –   Dagstuhl Seminar "Algorithms and Effectivity in Tropical Mathematics and Beyond", Nov. 28 - Dec. 02, 2016. Title of the talk "The equivalence between zero-sum games and tropical convexity".

- A. Hochart
    –   Game Theory PhD Seminar, Paris, February 1, 2016. Title of the talk: "Ergodic problems for zero-sum stochastic games".
    –   Conference SMAI-MODE, Toulouse, March 23-27, 2016. Title of the talk: "Une approche opérateur accrétif pour les jeux stochastiques avec critère ergodique".
    –   MTNS 2016 (22nd International Symposium on Mathematical Theory of Networks and Systems), Minneapolis, July 12-15, 2016, Title of the talk: "An Accretive Operator Approach to Ergodic Problems for Zero-Sum Games".

- M. MacCaig
    –   Birmingham Young Mathematicians Conference 2016. Title: "Tropical algebra: Optimisation, tropical polytopes and integer points".
    –   Student tropical algebraic geometry seminar (STAGS 2016), Yale. Title: "Calculating the volume of tropical polytopes is hard".
    –   ILAS 20th annual meeting, K.U.Leuven, July 11, 2016. Title: "Scaling to integer matrices in max-algebra".

– Dagstuhl Seminar on Algorithms and Effectivity in Tropical Mathematics.

- A. Niv
  – Conference "Recent advances in linear algebra and graph-theory", U.T.Chattanooga, March 5-6, 2016. Title of the talk: "Introduction to tropical total positivity".
  – Seminar at Afeka Academic College of Engineering, Tel-Aviv, May 8, 2016. Title of the talk: "Assignment problems via tropical matrices".
  – Tropical symposium, ILAS 20th annual meeting, K.U.Leuven, July 11, 2016. Title of the talk: "Total non-negativity via valuations in tropical algebra".

- M. Skomra
  – Conference SMAI-MODE, Toulouse, March 23–25, 2016. Title of the talk: "Les spectrahèdres tropicaux et leur relation aux jeux stochastiques".
  – Conference "Emerging Trends in Applied Mathematics and Mechanics" (ETAMM), Perpignan, May 30 - June 3, 2016. Title of the talk: "Nonarchimedean semidefinite programming and stochastic games".
  – Séminaire des doctorants du CMAP, Palaiseau, June 10, 2016. Title of the talk: "Une relation entre la programmation semi-définie paramétrique et les jeux stochastiques".
  – Conference of the International Linear Algebra Society (ILAS), Leuven, July 11-16, 2016. Title of the talk: "Nonarchimedean semidefinite programming and stochastic games".
  – International Symposium on Symbolic and Algebraic Computation (ISSAC), Waterloo, Ontario, Canada, July 19-22, 2016. Title of the talk: "Solving Generic Nonarchimedean Semidefinite Programs using Stochastic Game Algorithms".
  – Conference PGMO Days, EDF Labs Paris-Saclay, November 8-9, 2016. Title of the talk: "Solving Generic Nonarchimedean Semidefinite Programs using Stochastic Game Algorithms".
  – Dagstuhl Seminar "Algorithms and Effectivity in Tropical Mathematics and Beyond", Nov. 28 - Dec. 02, 2016. Title of the talk: "Tropical spectrahedra and stochastic mean payoff games".

- C. Walsh
  – Conference "New Methods in Finsler Geometry", Leipzig, July 5-9, 2016. Title of the talk: "Studying isometry groups using the horofunction boundary".

11. Bibliography

Major publications by the team in recent years

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[3] M. AKIAN, S. GAUBERT, R. NUSSBAUM. *Uniqueness of the fixed point of nonexpansive semidifferentiable maps*, in "Transactions of the American Mathematical Society", February 2016, vol. 368, n° 2, Also arXiv:1201.1536 [*DOI :* 10.1090/S0002-9947-2015-06413-7], https://hal.inria.fr/hal-00783682

[4] M. AKIAN, S. GAUBERT, C. WALSH. *The max-plus Martin boundary*, in "Doc. Math.", 2009, vol. 14, pp. 195–240

[5] X. ALLAMIGEON, P. BENCHIMOL, S. GAUBERT. *The Tropical Shadow-Vertex Algorithm Solves Mean Payoff Games in Polynomial Time on Average*, in "ICALP 2014", Lecture Notes in Computer Science, Springer, 2014, vol. 8572, pp. 89–100, http://dx.doi.org/10.1007/978-3-662-43948-7_8

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