



## Activity Report 2017

# Team TROPICAL

## TROPICAL

Inria teams are typically groups of researchers working on the definition of a common project, and objectives, with the goal to arrive at the creation of a project-team. Such project-teams may include other partners (universities or research institutions).

RESEARCH CENTER  
Saclay - Île-de-France

THEME  
Optimization and control of dynamic  
systems



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# Team TROPICAL

*Creation of the Team: 2016 January 01*

## Keywords:

### Computer Science and Digital Science:

- A2.4. - Verification, reliability, certification
- A6.2.5. - Numerical Linear Algebra
- A6.2.6. - Optimization
- A6.4.1. - Deterministic control
- A6.4.2. - Stochastic control
- A8.1. - Discrete mathematics, combinatorics
- A8.2. - Optimization
- A8.3. - Geometry, Topology
- A8.9. - Performance evaluation
- A8.11. - Game Theory

### Other Research Topics and Application Domains:

- B1.1.10. - Mathematical biology
- B4.4. - Energy delivery
- B4.4.1. - Smart grids
- B9.9. - Risk management

## 1. Personnel

### Research Scientists

- Stéphane Gaubert [Team leader, Inria, Senior Researcher, HDR]
- Marianne Akian [Inria, Senior Researcher, HDR]
- Xavier Allamigeon [Corps des Mines, under secondment, Inria, Senior Researcher]
- Cormac Walsh [Inria, Researcher]

### Post-Doctoral Fellows

- Daniel Jones [Inria, Post-Doctoral Fellow, from Aug 2017]
- Marie Maccaig [FMJH, Ecole polytechnique, Post-Doctoral Fellow, until Mar 2017]
- Aurelien Sagnier [Ecole polytechnique, from Sep 2017]

### PhD Students

- Vianney Boeuf [Ingénieur du corps des Ponts, ENPC, PhD Student, until Aug 2017]
- Jean-Bernard Eytard [Inria, PhD Student]
- Eric Fodjo [I-Fihn Consulting, Consultant, PhD Student]
- Paulin Jacquot [EDF (Cifre), PhD Student]
- Mateusz Skomra [Ecole polytechnique, PhD Student]
- Nikolas Stott [Inria, PhD Student]
- Duy Nghi, Benoît Tran [Univ Paris-Est Marne La Vallée, PhD Student, from Sep 2017]

### Technical staff

- Vasileios Charisopoulos [Inria, from May 2017 until Aug 2017]

### Administrative Assistants

- Hanadi Dib [Inria, from Sep 2017]

Corinne Petitot [Inria, until Sep 2017]

### Visiting Scientists

Zheng Hua [HKU, from Jun 2017 until Jul 2017]

Ricardo Katz [CONICET, from May 2017 until Jul 2017]

Gleb Koshevoy [Russian Academy of Sciences, from Feb 2017 until Mar 2017]

Zheng Qu [HKU, from May 2017 until Jul 2017]

## 2. Overall Objectives

### 2.1. Introduction

The project develops tropical methods motivated by applications arising in decision theory (deterministic and stochastic optimal control, game theory, optimization and operations research), in the analysis or control of classes of dynamical systems (including timed discrete event systems and positive systems), in the verification of programs and systems, and in the development of numerical algorithms. Tropical algebra tools are used in interaction with various methods, coming from convex analysis, Hamilton–Jacobi partial differential equations, metric geometry, Perron-Frobenius and nonlinear fixed-point theories, combinatorics or algorithmic complexity. The emphasis of the project is on mathematical modelling and computational aspects.

The subtitle of the *Tropical* project, namely, “structures, algorithms, and interactions”, refers to the spirit of our research, including a methodological component, computational aspects, and finally interactions with other scientific fields or real world applications, in particular through mathematical modelling.

### 2.2. Scientific context

Tropical algebra, geometry, and analysis have enjoyed spectacular development in recent years. Tropical structures initially arose to solve problems in performance evaluation of discrete event systems [61], combinatorial optimization [64], or automata theory [105]. They also arose in mathematical physics and asymptotic analysis [95], [92]. More recently, these structures have appeared in several areas of pure mathematics, in particular in the study of combinatorial aspects of algebraic geometry [84], [117], [107], [90], in algebraic combinatorics [78], and in arithmetics [68]. Also, further applications of tropical methods have appeared, including optimal control [99], program invariant computation [55] and timed systems verification [94], and zero-sum games [2].

The term ‘tropical’ generally refers to algebraic structures in which the laws originate from optimization processes. The prototypical tropical structure is the max-plus semifield, consisting of the real numbers, equipped with the maximum, thought of as an additive law, and the addition, thought of as a multiplicative law. Tropical objects appear as limits of classical objects along certain deformations (“log-limits sets” of Bergman, “Maslov dequantization”, or “Viro deformation”). For this reason, the introduction of tropical tools often yields new insights into old familiar problems, leading either to counterexamples or to new methods and results; see for instance [117], [101]. In some applications, like optimal control, discrete event systems, or static analysis of programs, tropical objects do not appear through a limit procedure, but more directly as a modelling or computation/analysis tool; see for instance [112], [61], [86], [65].

Tropical methods are linked to the fields of positive systems and of metric geometry [103], [11]. Indeed, tropically linear maps are monotone (a.k.a. order-preserving). They are also nonexpansive in certain natural metrics (sup-norm, Hopf oscillation, Hilbert’s projective metric, ...). In this way, tropical dynamical systems appear to be special cases of nonexpansive, positive, or monotone dynamical systems, which are studied as part of linear and non-linear Perron-Frobenius theory [93], [3]. Such dynamical systems are of fundamental importance in the study of repeated games [100]. Monotonicity properties are also essential in the understanding of the fixed points problems which determine program invariants by abstract interpretation [69]. The latter problems are actually somehow similar to the ones arising in the study of zero-sum games; see [7]. Moreover, positivity or monotonicity methods are useful in population dynamics, either in a discrete space setting [114]

or in a PDE setting [62]. In such cases, solving tropical problems often leads to solutions or combinatorial insights on classical problems involving positivity conditions (e.g., finding equilibria of dynamical systems with nonnegative coordinates, understanding the qualitative and quantitative behavior of growth rates / Floquet eigenvalues [9], etc). Other applications of Perron-Frobenius theory originate from quantum information and control [106], [111].

## 3. Research Program

### 3.1. Optimal control and zero-sum games

The dynamic programming approach allows one to analyze one or two-player dynamic decision problems by means of operators, or partial differential equations (Hamilton–Jacobi or Isaacs PDEs), describing the time evolution of the value function, i.e., of the optimal reward of one player, thought of as a function of the initial state and of the horizon. We work especially with problems having long or infinite horizon, modelled by stopping problems, or ergodic problems in which one optimizes a mean payoff per time unit. The determination of optimal strategies reduces to solving nonlinear fixed point equations, which are obtained either directly from discrete models, or after a discretization of a PDE.

**The geometry of solutions of optimal control and game problems** Basic questions include, especially for stationary or ergodic problems, the understanding of existence and uniqueness conditions for the solutions of dynamic programming equations, for instance in terms of controllability or ergodicity properties, and more generally the understanding of the structure of the full set of solutions of stationary Hamilton–Jacobi PDEs and of the set of optimal strategies. These issues are already challenging in the one-player deterministic case, which is an application of choice of tropical methods, since the Lax–Oleinik semigroup, i.e., the evolution semigroup of the Hamilton–Jacobi PDE, is a linear operator in the tropical sense. Recent progress in the deterministic case has been made by combining dynamical systems and PDE techniques (weak KAM theory [75]), and also using metric geometry ideas (abstract boundaries can be used to represent the sets of solutions [89], [4]). The two player case is challenging, owing to the lack of compactness of the analogue of the Lax–Oleinik semigroup and to a richer geometry. The conditions of solvability of ergodic problems for games (for instance, solvability of ergodic Isaacs PDEs), and the representation of solutions are only understood in special cases, for instance in the finite state space case, through tropical geometry and non-linear Perron-Frobenius methods [48],[12], [3].

**Algorithmic aspects: from combinatorial algorithms to the attenuation of the curse of dimensionality**

Our general goal is to push the limits of solvable models by means of fast algorithms adapted to large scale instances. Such instances arise from discrete problems, in which the state space may so large that it is only accessible through local oracles (for instance, in some web ranking applications, the number of states may be the number of web pages) [76]. They also arise from the discretization of PDEs, in which the number of states grows exponentially with the number of degrees of freedom, according to the “curse of dimensionality”. A first line of research is the development of *new approximation methods for the value function*. So far, classical approximations by linear combinations have been used, as well as approximation by suprema of linear or quadratic forms, which have been introduced in the setting of dual dynamic programming and of the so called “max-plus basis methods” [77]. We believe that more concise or more accurate approximations may be obtained by unifying these methods. Also, some max-plus basis methods have been shown to *attenuate the curse of dimensionality* for very special problems (for instance involving switching) [96], [80]. This suggests that the complexity of control or games problems may be measured by more subtle quantities than the mere number of states, for instance, by some forms of metric entropy (for example, certain large scale problems have a low complexity owing to the presence of decomposition properties, “highway hierarchies”, etc.). A second line of our research is the development of *combinatorial algorithms*, to solve large scale zero-sum two-player problems with discrete state space. This is related to current open problems in algorithmic game theory. In particular, the existence of polynomial-time algorithms for games with ergodic payment is an open question. See e.g. [5] for a polynomial time average complexity result derived by tropical methods. The

two lines of research are related, as the understanding of the geometry of solutions allows to develop better approximation or combinatorial algorithms.

### 3.2. Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

Several applications (including population dynamics [9] and discrete event systems [61], [67], [54]) lead to studying classes of dynamical systems with remarkable properties: preserving a cone, preserving an order, or being nonexpansive in a metric. These can be studied by techniques of non-linear Perron-Frobenius theory [3] or metric geometry [10]. Basic issues concern the existence and computation of the “escape rate” (which determines the throughput, the growth rate of the population), the characterizations of stationary regimes (non-linear fixed points), or the study of the dynamical properties (convergence to periodic orbits). Nonexpansive mappings also play a key role in the “operator approach” to zero-sum games, since the one-day operators of games are nonexpansive in several metrics, see [8].

### 3.3. Tropical algebra and convex geometry

The different applications mentioned in the other sections lead us to develop some basic research on tropical algebraic structures and in convex and discrete geometry, looking at objects or problems with a “piecewise-linear” structure. These include the geometry and algorithmics of tropical convex sets [56], [50], tropical semialgebraic sets [59], the study of semi-modules (analogues of vector spaces when the base field is replaced by a semi-field), the study of systems of equations linear in the tropical sense, investigating for instance the analogues of the notions of rank, the analogue of the eigenproblems [14], and more generally of systems of tropical polynomial equations. Our research also builds on, and concern, classical convex and discrete geometry methods.

### 3.4. Tropical methods applied to optimization, perturbation theory and matrix analysis

Tropical algebraic objects appear as a deformation of classical objects through various asymptotic procedures. A familiar example is the rule of asymptotic calculus,

$$e^{-a/\epsilon} + e^{-b/\epsilon} \asymp e^{-\min(a,b)/\epsilon}, \quad e^{-a/\epsilon} \times e^{-b/\epsilon} = e^{-(a+b)/\epsilon}, \quad (1)$$

when  $\epsilon \rightarrow 0^+$ . Deformations of this kind have been studied in different contexts: large deviations, zero-temperature limits, Maslov’s “dequantization method” [95], non-archimedean valuations, log-limit sets and Viro’s patchworking method [117], etc.

This entails a relation between classical algorithmic problems and tropical algorithmic problems, one may first solve the  $\epsilon = 0$  case (non-archimedean problem), which is sometimes easier, and then use the information gotten in this way to solve the  $\epsilon = 1$  (archimedean) case.

In particular, tropicalization establishes a connection between polynomial systems and piecewise affine systems that are somehow similar to the ones arising in game problems. It allows one to transfer results from the world of combinatorics to “classical” equations solving. We investigate the consequences of this correspondence on complexity and numerical issues. For instance, combinatorial problems can be solved in a robust way. Hence, situations in which the tropicalization is faithful lead to improved algorithms for classical problems. In particular, scalings for the polynomial eigenproblems based on tropical preprocessings have started to be used in matrix analysis [82], [85].

Moreover, the tropical approach has been recently applied to construct examples of linear programs in which the central path has an unexpectedly high total curvature [53], and it has also led to positive polynomial-time average case results concerning the complexity of mean payoff games. Similarly, we are studying semidefinite programming over non-archimedean fields [59], [58], with the goal to better understand complexity issues in classical semidefinite and semi-algebraic programming.



## 4. Application Domains

### 4.1. Discrete event systems (manufacturing systems, networks)

One important class of applications of max-plus algebra comes from discrete event dynamical systems [61]. In particular, modelling timed systems subject to synchronization and concurrency phenomena leads to studying dynamical systems that are non-smooth, but which have remarkable structural properties (nonexpansiveness in certain metrics, monotonicity) or combinatorial properties. Algebraic methods allow one to obtain analytical expressions for performance measures (throughput, waiting time, etc). A recent application, to emergency call centers, can be found in [54].

### 4.2. Optimal control and games

Optimal control and game theory have numerous well established applications fields: mathematical economy and finance, stock optimization, optimization of networks, decision making, etc. In most of these applications, one needs either to derive analytical or qualitative properties of solutions, or design exact or approximation algorithms adapted to large scale problems.

### 4.3. Operations Research

We develop, or have developed, several aspects of operations research, including the application of stochastic control to optimal pricing, optimal measurement in networks [108]. Applications of tropical methods arise in particular from discrete optimization [65], [66], scheduling problems with and-or constraints [102], or product mix auctions [116].

### 4.4. Computing program and dynamical systems invariants

A number of programs and systems verification questions, in which safety considerations are involved, reduce to computing invariant subsets of dynamical systems. This approach appears in various guises in computer science, for instance in static analysis of program by abstract interpretation, along the lines of P. and R. Cousot [69], but also in control (eg, computing safety regions by solving Isaacs PDEs). These invariant sets are often sought in some tractable effective class: ellipsoids, polyhedra, parametric classes of polyhedra with a controlled complexity (the so called “templates” introduced by Sankaranarayanan, Sipma and Manna [109]), shadows of sets represented by linear matrix inequalities, disjunctive constraints represented by tropical polyhedra [55], etc. The computation of invariants boils down to solving large scale fixed point problems. The latter are of the same nature as the ones encountered in the theory of zero-sum games, and so, the techniques developed in the previous research directions (especially methods of monotonicity, nonexpansiveness, discretization of PDEs, etc) apply to the present setting, see e.g. [79], [83] for the application of policy iteration type algorithms, or for the application for fixed point problems over the space of quadratic forms [7]. The problem of computation of invariants is indeed a key issue needing the methods of several fields: convex and nonconvex programming, semidefinite programming and symbolic computation (to handle semialgebraic invariants), nonlinear fixed point theory, approximation theory, tropical methods (to handle disjunctions), and formal proof (to certify numerical invariants or inequalities).

## 5. Highlights of the Year

### 5.1. Highlights of the Year

#### 5.1.1. Performance evaluation of the 17-18-112 call center in Paris

Vianney Bœuf completed his PhD, done in collaboration with Brigade des Sapeurs Pompiers de Paris, on the performance evaluation of the new organization of the Paris emergency call center developed by Préfecture de Police. See Section 7.5.2.

### 5.1.2. Maximal upper bounds in Löwner order

A classical theorem of Kadison (1951) shows that the set of real quadratic forms, equipped with the pointwise order, is an antilattice, meaning that two quadratic forms have a least upper bound (or dually, a greatest lower bound) if and only if they are comparable. In [23], Nikolas Stott gave a quantitative version of Kadison theorem, characterizing the set of minimal upper bound as the quotient an indefinite orthogonal group. Applications of these ideas to hybrid systems verification appeared in [16], [30].

### 5.1.3. Formal proofs in linear programming

Xavier Allamigeon and Ricardo Katz have formalized in the proof assistant Coq several basic results in the theory of convex polyhedra and linear optimization. These include Farkas Lemma, the duality theorem of linear programming, separation from convex hulls, Minkowski Theorem, etc. See [27] and Section 7.3.1.

## 6. New Software and Platforms

### 6.1. Coq-Polyhedra

KEYWORDS: Coq - Polyhedra - Automated theorem proving - Linear optimization

SCIENTIFIC DESCRIPTION: Coq-Polyhedra is a library providing a formalization of convex polyhedra in the Coq proof assistant. While still in active development, it provides an implementation of the simplex method, and already handles the basic properties of polyhedra such as emptiness, boundedness, membership. Several fundamental results in the theory of convex polyhedra, such as Farkas Lemma, duality theorem of linear programming, and Minkowski Theorem, are also formally proved.

The formalization is based on the Mathematical Components library, and makes an extensive use of the boolean reflection methodology.

FUNCTIONAL DESCRIPTION: Coq-Polyhedra is a library which aims at formalizing convex polyhedra in Coq

- Participants: Xavier Allamigeon, Vasileios Charisopoulos and Ricardo Katz
- Partner: CIFASIS
- Contact: Xavier Allamigeon
- Publication: [A Formalization of Convex Polyhedra Based on the Simplex Method](#)
- URL: <https://github.com/nhojem/Coq-Polyhedra>

## 7. New Results

### 7.1. Optimal control and zero-sum games

#### 7.1.1. Fixed points of order preserving homogeneous maps and zero-sum games

**Participants:** Marianne Akian, Stéphane Gaubert.

The PhD work of Antoine Hochart [88] was dealing with the applications of methods of non-linear fixed point theory to zero-sum games.

A highlight of his PhD is the characterization of the property of ergodicity for zero-sum games. In the special “zero-player” case, i.e., for a Markov chain equipped with an additive functional (payment) of the trajectory, the ergodicity condition entails that the mean payoff is independent of the initial state, for any choice of the payment. In the case of finite Markov chains, ergodicity admits several characterizations, including a combinatorial one (the uniqueness of the final class). This carries over to the two player case: ergodicity is now characterized by the absence of certain pairs of conjugate invariant sets (dominions), and it can be checked using directed hypergraphs algorithms. This leads to an explicit combinatorial sufficient condition for the solvability of the “ergodic equation”, which is the main tool in the numerical approach of the mean payoff problem. These results appeared in [52] for the case of bounded paiements. A more general approach was developed in [87], in which zero-sum games are now studied abstractly in terms of accretive operators. This allows one to show that the bias vector (the solution of the ergodic equation) is unique for a generic perturbation of the payments. A more recent work include the introduction of an abstract game allowing us to deal with general monotone additively homogeneous operators and thus to unbounded paiements.

Another series of results of the thesis concern the finite action space, showing that the set of payments for which the bias vector is not unique coincides with the union of lower dimensional cells of a polyhedral complex, which an application to perturbation schemes in policy iteration [12].

A last result of the thesis is a representation theorem for “payment free” Shapley operators, showing that these are characterized by monotonicity and homogeneity axioms [13]. This extends to the two-player case known representation theorems for risk measures.

### 7.1.2. *The operator approach to entropy games*

**Participants:** Marianne Akian, Stéphane Gaubert.

Entropy games were recently introduced by Asarin et al. A player (Despot) wishes to minimize a measure of “freedom” given by a topological entropy, whereas the other player (Tribune) wishes to maximize it. In [25], we developed an operator approach for entropy games. We showed that they reduce to risk sensitive type game problems, and deduced that entropy games in Despot has a few positions with non-trivial actions can be solved in polynomial time.

### 7.1.3. *Probabilistic and max-plus approximation of Hamilton-Jacobi-Bellman equations*

**Participants:** Marianne Akian, Eric Fodjo.

The PhD thesis of Eric Fodjo concerns stochastic control problems obtained in particular in the modelisation of portfolio selection with transaction costs. The dynamic programming method leads to a Hamilton-Jacobi-Bellman partial differential equation, on a space with a dimension at least equal to the number of risky assets. The curse of dimensionality does not allow one to solve numerically these equations for a large dimension (greater to 5). We propose to tackle these problems with numerical methods combining policy iterations, probabilistic discretisations, max-plus discretisations, in order to increase the possible dimension.

We consider fully nonlinear Hamilton-Jacobi-Bellman equations associated to diffusion control problems with finite horizon involving a finite set-valued (or switching) control and possibly a continuum-valued control. In [47], we constructed a lower complexity probabilistic numerical algorithm by combining the idempotent expansion properties obtained by McEneaney, Kaise and Han [91], [97] for solving such problems with a numerical probabilistic method such as the one proposed by Fahim, Touzi and Warin [74] for solving some fully nonlinear parabolic partial differential equations, when the volatility does not oscillate too much. In [38], [39] (also presented in [24]), we improve the method of Fahim, Touzi and Warin by introducing probabilistic schemes which are monotone without any restrictive condition, allowing one to solve fully nonlinear parabolic partial differential equations with general volatilities. We study the convergence and obtain error estimates when the parameters and the value function are bounded. We are now studying the more general quadratic growth case.

### 7.1.4. *Tropical-SDDP algorithms for stochastic control problems involving a switching control*

**Participants:** Marianne Akian, Duy Nghi, Benoît Tran.

The PhD thesis of Benoît Tran, supervised by Jean-Philippe Chancelier (ENPC) and Marianne Akian concerns the numerical solution of the dynamic programming equation of discrete time stochastic control problems.

Several methods have been proposed in the literature to bypass the curse of dimensionality difficulty of such an equation, by assuming a certain structure of the problem. Examples are the max-plus based method of McEneaney [98], [99], the stochastic dual dynamic programming (SDDP) algorithm of Pereira and Pinto [104], the mixed integer dynamic approximation scheme of Philpott, Faisal and Bonnans [60], the probabilistic numerical method of Fahim, Touzi and Warin [74]. We propose to associate and compare these methods in order to solve more general structures, in particular problems involving a finite set-valued (or switching) control and a continuum-valued control, with the property that the value function associated to a fixed switching strategy is convex.

## 7.2. Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

### 7.2.1. Order reversing maps on cones

**Participant:** Cormac Walsh.

We have been studying non-linear operators on open cones, particularly ones that preserve or reverse the order structure associated to the cone. A bijective map that preserves the order in both directions is called an order isomorphism. Those that reverse the order in both directions are order antimorphisms. These are closely related to the isometries of the Hilbert and Thompson metrics on the cone.

Previously, we have shown [118] that if there exists an antimorphism on an open cone that is homogeneous of degree  $-1$ , then the cone must be a symmetric cone, that is, have a transitive group of linear automorphisms and be self-dual.

The technique was to consider the Funk metric on the cone, which is a non-symmetric metric defined using the order and homogeneity structures. Each antimorphism on a cone that is homogeneous of degree  $-1$  reverses this metric, and so interchanges the two horofunction boundaries of the metric, the one in the forward direction and the one in the backward direction. By studying these boundaries we obtained the result.

More recently, we have shown [45] that the homogeneity assumption is not actually necessary: every antimorphism on a cone is automatically homogeneous of degree  $-1$ .

Without the homogeneity assumption, the metric techniques do not work. Instead, it was necessary to study how the map acts on line segments parallel to extreme rays of the cone. This is similar to the what was done by Rothaus, Noll and Schäffer, and Artstein-Avidan and Slomka in their work on order isomorphisms. This means that our proof is essentially finite dimensional. Indeed, there are many interesting cones in infinite dimension that have few or no extreme rays.

In infinite dimension, it is natural to consider order unit spaces as a generalisation of cones, and JB-algebras as a generalisation of symmetric cones. Lemmens, Roelands, and Wortel have asked whether the existence of an order antimorphism that is homogeneous of degree  $-1$  on the cone of an order-unit space implies that the space is a JB-algebra? The result above suggests further that one might even be able to drop the homogeneity assumption.

### 7.2.2. The set of minimal upper bounds of two matrices in the Loewner order

**Participant:** Nikolas Stott.

A classical theorem of Kadison shows that the space of symmetric matrices equipped with the Loewner order is an anti-lattice, meaning that two matrices have a least upper bound if and only if they are comparable. In [115], we refined this theorem by characterizing the set of minimal upper bounds: we showed that it is homeomorphic to the quotient space  $O(p) \setminus O(p, q) / O(q)$ , where  $O(p, q)$  denotes the orthogonal group associated to the quadratic form with signature  $(p, q)$ , and  $O(p)$  denotes the standard  $p$ th orthogonal group.

### 7.2.3. Checking the strict positivity of Kraus maps is NP-hard

**Participant:** Stéphane Gaubert.

In collaboration with Zheng Qu (now with HKU, Hong Kong), I studied several decision problems arising from the spectral theory of Kraus maps (trace preserving completely positive maps), acting on the cone of positive semidefinite matrices. The latter appear in quantum information. We showed that checking the irreducibility (absence of non-trivial invariant face of the cone) and primitivity properties (requiring the iterates of the map to send the cone to its interior) can be checked in polynomial time, whereas checking positivity (whether the map sends the cone to its interior) is NP-hard. In [20], we studied complexity issues related to Kraus maps, and showed in particular that checking whether a Kraus map sends the cone to its interior is NP-hard.

## 7.3. Tropical algebra and convex geometry

### 7.3.1. Formalizing convex polyhedra in Coq

**Participants:** Xavier Allamigeon, Ricardo Katz [Conicet, Argentine].

In [27], we have made the first steps of a formalization of the theory of convex polyhedra in the proof assistant Coq. The originality of our approach lies in the fact that our formalization is carried out in an effective way, in the sense that the basic predicates over polyhedra (emptiness, boundedness, membership, etc) are defined by means of Coq programs. All these predicates are then proven to correspond to the usual logical statements. The latter take the form of the existence of certificates: for instance, the emptiness of a polyhedron is shown to be equivalent to the existence of a certificate *a la* Farkas. This equivalence between Boolean predicates and formulas living in the kind Prop is implemented by using the boolean reflection methodology, and the supporting tools provided by the Mathematical Components library and its tactic language. The benefit of the effective nature of our approach is demonstrated by the fact that we easily arrive at the proof of important results on polyhedra, such as several versions of Farkas Lemma, duality theorem of linear programming, separation from convex hulls, Minkowski Theorem, etc.

Our effective approach is made possible by implementing the simplex method inside Coq, and proving its correctness and termination. Two difficulties need to be overcome to formalize it. On the one hand, we need to deal with its termination. More precisely, the simplex method iterates over the so-called bases. Its termination depends on the specification of a pivoting rule, whose aim is to determine, at each iteration, the next basis. In this work, we have focused on proving that the lexicographic rule ensures termination. On the other hand, the simplex method is actually composed of two parts. The part that we previously described, called Phase II, requires an initial basis to start with. Finding such a basis is the purpose of Phase I. It consists in building an extended problem (having a trivial initial basis), and applying to it Phase II. Both phases need to be formalized to obtain a fully functional algorithm. a

### 7.3.2. Tropical totally positive matrices

**Participant:** Stéphane Gaubert.

In [81] (joint work with Adi Niv) we investigate the tropical analogues of totally positive and totally non-negative matrices, i.e, the images by the valuation of the corresponding classes of matrices over a non-archimedean field. We show in particular that tropical totally positive matrices essentially coincide with the Monge matrices (defined by the positivity of  $2 \times 2$  tropical minors), arising in optimal transport. More recent developments include relations between tropical total positivity and planar networks.

### 7.3.3. Tropical compound matrix identities

**Participants:** Marianne Akian, Stéphane Gaubert.

A number of polynomial identities in tropical semirings can be derived from their classical analogues by application of a transfer principle [49], [51]. In the present work [40], joint with Adi Niv, we prove identities on compound matrices in extended tropical semirings, which cannot be obtained by transfer principles, but are rather obtained by combinatorial methods. Such identities include analogues to properties of conjugate matrices, powers of matrices and  $\text{adj}(A) \det(A)^{-1}$ , all of which have implications on the eigenvalues of the corresponding matrices. A tropical Sylvester-Franke identity is provided as well.

### 7.3.4. Group algebra in characteristic one and invariant distances over finite groups

**Participant:** Stéphane Gaubert.

In [19] (joint work with Dominique Castella), we investigated a tropical analogue of group algebras. We studied tropical characters and related them to invariant distances over groups.

### 7.3.5. Volume and integer points of tropical polytopes

**Participants:** Marie Maccaig, Stéphane Gaubert.

We investigate in [43] the volume of tropical polytopes, as well as the number of integer points contained in integer polytopes. We proved that even approximating these values for a tropical polytope given by its vertices is hard, with no approximation algorithm with factor  $2^{\text{poly}(m,n)}$  existing unless  $P = NP$ . We further proved the  $\#P$ -hardness for the analogous problems for tropical polytopes instead defined by inequalities.

## 7.4. Tropical methods applied to optimization, perturbation theory and matrix analysis

### 7.4.1. Majorization inequalities for valuations of eigenvalues using tropical algebra

**Participants:** Marianne Akian, Stéphane Gaubert.

In [14], with Meisam Sharify (IPM, Tehran, Iran), we establish log-majorization inequalities of the eigenvalues of matrix polynomials using the tropical roots of some scalar polynomials depending only on the norms of the matrix coefficients. This extends to the case of matrix polynomials some bounds obtained by Hadamard, Ostrowski and Pólya for the roots of scalar polynomials.

These works have been presented in [46].

### 7.4.2. Tropicalization of the central path and application to the complexity of interior point methods

**Participants:** Xavier Allamigeon, Stéphane Gaubert.

This work is in collaboration with Pascal Benchimol (EDF Labs) and Michael Joswig (TU Berlin).

In optimization, path-following interior point methods are driven to an optimal solution along a trajectory called the central path. The *central path* of a linear program  $\text{LP}(A, b, c) \equiv \min\{c \cdot x \mid Ax \leq b, x \geq 0\}$  is defined as the set of the optimal solutions  $(x^\mu, w^\mu)$  of the barrier problems:

$$\begin{aligned} \text{minimize} \quad & c \cdot x - \mu \left( \sum_{j=1}^n \log x_j + \sum_{i=1}^m \log w_i \right) \\ \text{subject to} \quad & Ax + w = b, \quad x > 0, \quad w > 0 \end{aligned}$$

While the complexity of interior point methods is known to be polynomial, an important question is to study the number of iterations which are performed by interior point methods, in particular whether it can be bounded by a polynomial in the dimension  $(mn)$  of the problem. This is motivated by Smale 9th problem [113], on the existence of a strongly polynomial complexity algorithm for linear programming. So far, this question has been essentially addressed through the study of the curvature of the central path, which measures how far a path differs from a straight line, see [71], [70], [73], [72]. In particular, by analogy with the classical Hirsch conjecture, Deza, Terlaky and Zinchenko [72] proposed the “continuous analogue of the Hirsch conjecture”, which says that the total curvature of the central path is linearly bounded in the number  $m$  of constraints.

In a work of X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig [41], we prove that primal-dual log-barrier interior point methods are not strongly polynomial, by constructing a family of linear programs with  $3r + 1$  inequalities in dimension  $2r$  for which the number of iterations performed is in  $\Omega(2^r)$ . The total curvature of the central path of these linear programs is also exponential in  $r$ , disproving the continuous analogue of the Hirsch conjecture.

Our method is to tropicalize the central path in linear programming. The tropical central path is the piecewise-linear limit of the central paths of parameterized families of classical linear programs viewed through logarithmic glasses. We give an explicit geometric characterization of the tropical central path, as a tropical analogue of the barycenter of a sublevel set of the feasible set induced by the duality gap. We study the convergence properties of the classical central path to the tropical one. This allows us to show that the number of iterations performed by interior point methods is bounded from below by the number of tropical segments constituting the tropical central path.

### 7.4.3. Tropical approach to semidefinite programming

**Participants:** Xavier Allamigeon, Stéphane Gaubert, Mateusz Skomra.

Semidefinite programming consists in optimizing a linear function over a spectrahedron. The latter is a subset of  $\mathbb{R}^n$  defined by linear matrix inequalities, i.e., a set of the form

$$\left\{ x \in \mathbb{R}^n : Q^{(0)} + x_1 Q^{(1)} + \dots + x_n Q^{(n)} \succeq 0 \right\}$$

where the  $Q^{(k)}$  are symmetric matrices of order  $m$ , and  $\succeq$  denotes the Loewner order on the space of symmetric matrices. By definition,  $X \succeq Y$  if and only if  $X - Y$  is positive semidefinite.

Semidefinite programming is a fundamental tool in convex optimization. It is used to solve various applications from engineering sciences, and also to obtain approximate solutions or bounds for hard problems arising in combinatorial optimization and semialgebraic optimization.

A general issue in computational optimization is to develop combinatorial algorithms for semidefinite programming. Indeed, semidefinite programs are usually solved via interior point methods. However, the latter provide an approximate solution in a polynomial number of iterations, provided that a strictly feasible initial solution. Semidefinite programming becomes a much harder matter if one requires an exact solution. The feasibility problem belongs to  $\text{NP}_{\mathbb{R}} \cap \text{coNP}_{\mathbb{R}}$ , where the subscript  $\mathbb{R}$  refers to the BSS model of computation. It is not known to be in NP in the bit model.

We address semidefinite programming in the case where the field  $\mathbb{R}$  is replaced by a nonarchimedean field, like the field of Puiseux series. In this case, methods from tropical geometry can be applied and are expected to allow one, in generic situations, to reduce semialgebraic problems to combinatorial problems, involving only the nonarchimedean valuations (leading exponents) of the coefficients of the input.

To this purpose, we first study tropical spectrahedra, which are defined as the images by the valuation of nonarchimedean spectrahedra. We establish that they are closed semilinear sets, and that, under a genericity condition, they are described by explicit inequalities expressing the nonnegativity of tropical minors of order 1 and 2. These results are gathered in the preprint [59].

Then, we show in [17] that the feasibility problem for a generic tropical spectrahedron is equivalent to solving a stochastic mean payoff game (with perfect information). The complexity of these games is a long-standing open problem. They are not known to be polynomial, however they belong to the class  $\text{NP} \cap \text{coNP}$ , and they can be solved efficiently in practice. This allows to apply stochastic game algorithms to solve nonarchimedean semidefinite feasibility problems. We obtain in this way both theoretical bounds and a practicable method which solves some large scale instances.

A long-standing problem is to characterize the convex semialgebraic sets that are SDP representable, meaning that they can be represented as the image of a spectrahedron by a (linear) projector. Helton and Nie conjectured that every convex semialgebraic set over the field of real numbers are SDP representable. Recently, [110] disproved this conjecture. In [26], we show, however, that the following result, which may be thought of as a tropical analogue of this conjecture, is true: over a real closed nonarchimedean field of Puiseux series, the convex semialgebraic sets and the projections of spectrahedra have precisely the same images by the nonarchimedean valuation. The proof relies on game theory methods applied to our previous results [59] and [17].

## 7.5. Applications

### 7.5.1. *Geometry of the Loewner order and application to the synthesis of quadratic invariants in static analysis of program*

**Participants:** Xavier Allamigeon, Stéphane Gaubert, Nikolas Stott.

This section presents the PhD work of Nikolas Stott. An essential part of the present work is in collaboration with Éric Goubault and Sylvie Putot (from LIX).

We develop a numerical abstract domain based on ellipsoids designed for the formal verification of switched linear systems. The novelty of this domain does not consist in the use of ellipsoids as abstractions, but rather in the fact that we overcome two key difficulties which so far have limited the use of ellipsoids in abstract interpretation. The first issue is that the ordered set of ellipsoids does not constitute a lattice. This implies that there is a priori no canonical choice of the abstraction of the union of two sets, making the analysis less predictable as it relies on the selection of good upper bounds. The second issue is that most recent works using on ellipsoids rely on LMI methods. The latter are efficient on moderate size examples but they are inherently limited by the complexity of interior point algorithms, which, in the case of matrix inequality problems, do not scale as well as for linear programming or second order cone programming problems.

We developed a new approach, in which we reduce the computation of an invariant to the determination of a fixed point, or eigenvector, of a non-linear map that provides a safe upper-approximation of the action induced by the program on the space of quadratic forms. This allows one to obtain invariants of systems of sized inaccessible by LMI methods, at the price of a limited loss of precision. A key ingredient here is the fast computation of least upper bounds in Löwner ordering, by an algebraic algorithm. This relies on the study of the geometry of the space of quadratic forms (Section 7.2.2).

The initial part of this work was described in the article [57], in which we obtained a single ellipsoidal invariant. In [16], we showed that finer disjunctive invariants, expressed as union of ellipsoids, can still be obtained by nonlinear fixed point methods in a scalable way. In [30], we developed a dual approach, which we applied to the problem of computing the joint spectral radius. We showed that an approximation of a Barabanov norm by a supremum of quadratic forms can be obtained by solving a nonlinear eigenvalue problem involving “tropical Kraus maps”. The latter can be thought of as the tropical analogues of the completely positive maps appearing in quantum information. The fixed point methods in [30] allow one to solve large scale instances, unaccessible by earlier (LMI based) methods.

### 7.5.2. *Performance evaluation of an emergency call center*

**Participants:** Xavier Allamigeon, Vianney Boeuf, Stéphane Gaubert.

This work arose from a question raised by Régis Reboul from Préfecture de Police de Paris (PP), regarding the analysis of the projected evolution of the treatment of emergency calls (17-18-112). This work benefited from the help of LtL Stéphane Raclot, from Brigade de Sapeurs de Pompiers de Paris (BSPP), now with PP. It is part of the PhD work of Vianney Bœuf, carried out in collaboration with BSPP.

We introduced an algebraic approach which allows to analyze the performance of systems involving priorities and modeled by timed Petri nets. Our results apply to the class of Petri nets in which the places can be partitioned in two categories: the routing in certain places is subject to priority rules, whereas the routing at the other places is free choice.

We initially introduced a discrete model in [54], showing that the counter variables, which determine the number of firings of the different transitions as a function of time, are the solutions of a piecewise linear dynamical system. We showed the stationary regimes are precisely the solutions of a set of lexicographic piecewise linear equations, which constitutes a polynomial system over a tropical (min-plus) semifield of germs. However, the convergence to a stationary regime may not occur in the discrete model. We developed in [15] a continuous time analogue of this model, involving a piecewise linear dynamical systems, and showed that it has the same stationary regimes, avoiding some pathologies of the discrete model.



In essence, this result shows that computing stationary regimes reduces to solving tropical polynomial systems. Solving tropical polynomial systems is one of the most basic problems of tropical geometry. The latter provides insights on the nature of solutions, as well as algorithmic tools. In particular, the tropical approach allows one to determine the different congestion phases of the system. This analysis has been recovered by a probabilistic model in [42].

We applied this approach to a case study relative to the project led by Préfecture de Police de Paris, involving BSPP, of a new organization to handle emergency calls to Police (number 17), Firemen (number 18), and untyped emergency calls (number 112), in the Paris area. We combined explicit analytic computations of the different congestion phases of a simplified model and extensive simulations of a realistic and detailed model, to evaluate the performance of the center as a function of the number of operators. This analysis also suggested some ways to monitor early signs of potential congestions as well as possible correcting measures to avoid congestion.

### 7.5.3. *Tropical models of fire propagation in urban areas*

**Participants:** Stéphane Gaubert, Daniel Jones.

As part of the team work in the ANR project Democrite, we developed a model of fire propagation in urban areas, involving a discrete analogue of a Hamilton-Jacobi PDE. This model indicates that the fire propagates according to polyhedral ball, which is in accordance from data from historical fires (London, Chicago, or more recently Kobe).

### 7.5.4. *Smart Data Pricing*

**Participants:** Marianne Akian, Jean-Bernard Eytard.

This work is in collaboration with Mustapha Bouhtou (Orange Labs).

The PhD work of Jean-Bernard Eytard concerns the optimal pricing of data traffic in mobile networks. We developed a bilevel programming approach, allowing to an operator to balance the load in the network through price incentives. We showed that a subclass of bilevel programs can be solved in polynomial time, by combining methods of tropical geometry and of discrete convexity. This work is presented in [28]. In a followup work (collaboration with Gleb Koshevoy), we managed to extend these results to wider classes of bilevel problems, and to relate them to competitive equilibria problems.

### 7.5.5. *Game theory models of decentralized mechanisms of pricing of the smart grid*

**Participants:** Stéphane Gaubert, Paulin Jacquot.

This work is in collaboration with Nadia Oudjane and Olivier Beaude (EDF).

The PhD work of Paulin Jacquot concerns the application of game theory techniques to pricing of energy. We are developing a game theory framework for demand side management in the smart grid, in which users have movable demands (like charging an electric vehicle). We compared in particular the daily and hourly billing mechanisms. The latter, albeit more complex to analyse, has a merit in terms of incentives, as it leads the user to move his or her consumption at off peak hours. We showed the Nash equilibrium is unique, under some assumptions, and gave theoretical bounds of the price of anarchy of the game with a hourly billing, showing this mechanism remains efficient while being more “fair” than the daily billing. We proposed and tested decentralized algorithms to compute the Nash equilibrium. These contributions are presented in [31], [32], [44].

## 8. Bilateral Contracts and Grants with Industry

### 8.1. Bilateral Contracts with Industry

- Yield management methods applied to the pricing of data traffic in mobile networks. CRE (research contract) with Orange Labs (Orange Labs partner: Mustapha Bouhtou).

- Decentralized mechanisms of operation of power systems: equilibria and efficiency. A collaboration started on this topic at the fall, Nadia Oudjane, Olivier Beaudé, and Riadh Zorgati from EDF-labs. This leads to the PhD work of Paulin Jacquot, supervised by Stéphane Gaubert (CIFRE PhD).

## 9. Partnerships and Cooperations

### 9.1. National Initiatives

#### 9.1.1. ANR

- Projet ANR MALTHY (Méthodes ALgébriques pour la vérification de modèles Temporisés et HYbrides), responsable T. Dang. Partenaires : Verimag, CEA LIST, Inria Rennes, Inria Saclay, VISEO/Object Direct.
- Projet ANR DEMOCRITE ("DEmonstrateur d'un MOteur de Couverture des Risques sur un TERRitoire), responsable Emmanuel Lapébie (CEA). Partenaires : CEA-GRAMAT, BSPP, Inria Saclay (Maxplus), Institut PPRIME - UPR3346 (CNRS, Univ. Poitiers, ISAE-ENSMA), IPSIS, SYSTEL, ARMINES-E.M. Alès-ISR, CERDACC (Univ. de Haute-Alsace).

#### 9.1.2. Programme Gaspard Monge pour l'Optimisation

- Projet intitulé "Méthodes tropicales pour l'optimisation", responsable X. Allamigeon, faisant intervenir M. Akian, V. Boeuf, S. Gaubert, A. Hochart, R. Katz, et M. Skomra.

### 9.2. International Initiatives

#### 9.2.1. Inria International Partners

##### 9.2.1.1. Informal International Partners

- Collaboration with Ricardo D. Katz, CIFASIS-CONICET, Rosario (Argentina). Research invitation at CMAP during 2 months.

#### 9.2.2. Participation in International Programs

- Collaboration with Gleb Koshevoy, Poncelet Laboratory, Moscow (research invitation of Gleb Koshevoy at CMAP during 2 months, research invitation of Stéphane Gaubert at Poncelet Laboratory during 1 week).

### 9.3. International Research Visitors

#### 9.3.1. Visits of International Scientists

- Gleb Koshevoy (Russian Academy of Sciences), Feb-March, 2017.
- Shmuel Friedland (University of Illinois at Chicago), one week in May 2017.
- Zheng Qu (Hong Kong University), June-July 2017
- Zheng Hua (Hong Kong University), June-July 2017
- Rajendra Bhatia (Indian Statistical Institute, New Delhi), 1 week in Dec 2017.
- Floris Claassens (University of Kent), 1 week in Dec 2017.

## 10. Dissemination

### 10.1. Promoting Scientific Activities

#### 10.1.1. Scientific Events Organisation

##### 10.1.1.1. General Chair, Scientific Chair

- Stéphane Gaubert is the coordinator of the Gaspard Monge Program for Optimization and Operations Research, a corporate sponsorship program, operated by Fondation Mathématique Jacques Hadamard, supported by EDF, Orange and Thales, see <https://www.fondation-hadamard.fr/fr/pgmo/>

#### 10.1.1.2. Member of the Organizing Committees

- S. Gaubert co-organizes the “Séminaire Parisien d’Optimisation”.

### 10.1.2. Scientific Events Selection

#### 10.1.2.1. Chair of Conference Program Committees

- S. Gaubert, Chair of the PGMO days, EDF Labs Paris Saclay, Nov 13-14, 2017. <https://www.fondation-hadamard.fr/fr/pgmo/pgmodays>

### 10.1.3. Journal

#### 10.1.3.1. Member of the Editorial Boards

- S. Gaubert is member of the editorial committee of the collection Mathématiques et Applications, SMAI and Springer.
- S. Gaubert is associate editor of Linear and Multilinear Algebra.
- S. Gaubert is associate editor of RAIRO Operations research.

### 10.1.4. Invited Talks

- S. Gaubert
  - Noncommutative geometry: number theory, celebration of Alain Connes’ 70th birthday, Shanghai, March 23 - April 7, 2017. Tropical modules, zero-sum games and non-archimedean optimization.
  - Mathematical morphology and its applications to image and signal processing Fontainebleau, Fontainebleau, May 15 – May 17, 2017. Tropical and non-linear Perron-Frobenius methods for optimal control and zero-sum games

### 10.1.5. Research Administration

- M. Akian :
  - Member of the “comité de liaison SMAI-MODE” since June 2015.
  - Member of the laboratory council of CMAP.
- S. Gaubert :
  - Member of the scientific council of CMAP.
- X. Allamigeon:
  - Member of the scientific committee of Inria Saclay – Ile-de-France.
  - Member of the Applied Mathematics Department committee at Ecole Polytechnique.

## 10.2. Teaching - Supervision - Juries

### 10.2.1. Teaching

- M. Akian
  - Course “Markov decision processes: dynamic programming and applications” joint between (3rd year of) ENSTA and M2 “Mathématiques et Applications”, U. Paris Saclay, “Optimization”, and shared with Jean-Philippe Chancelier (ENPC), 15 hours each.
- X. Allamigeon

- Petites classes et encadrement d’enseignements d’approfondissement de Recherche Opérationnelle en troisième année à l’École Polytechnique (programme d’approfondissement de Mathématiques Appliquées) (niveau M1).
- Cours du M2 “Optimisation” de l’Université Paris Saclay, cours partagé avec Manuel Ruiz (RTE) et Dominique Quadri (LRI, Université Paris Sud).
- Co-responsabilité du programme d’approfondissement en mathématiques appliquées (troisième année) à l’École Polytechnique.
- V. Boeuf
  - Petite classe du cours de tronc commun de 1ere année "Introduction à l’optimisation" de l’École des ponts (ENPC), niveau L3.
- J.B. Eytard
  - Cours de niveau L1-L2 à l’IUT d’Informatique d’Orsay (Univ. Paris-Sud), dans le cadre d’un monitorat (64h) (théorie des graphes, recherche opérationnelle, modélisation mathématique).
- S. Gaubert
  - Course “Systèmes à Événements Discrets”, option MAREVA, ENSMP.
  - Course “Algèbre max-plus pour le contrôle optimal et les jeux” of “Parcours Optimisation, Jeux et Dynamique” (ODJ) of M2 “Mathématiques et Applications” of Paris 6 University and École Polytechnique.
  - Lecture of Operations Research, third year of École Polytechnique. The lectures notes were published as a book [63].
- M. Skomra
  - TD de mathématiques à l’UPMC.
- N. Stott
  - Cours et TD “Introduction à la programmation graphique en C++”, option MAREVA et semaine Athens à l’ENSMP (École des Mines de Paris), niveau M1.

### 10.2.2. Supervision

- PhD in progress : Eric Fodjo, registered at École Polytechnique, since October 2013, thesis supervisor: Marianne Akian.
- PhD : Nikolas Stott, registered at École Polytechnique, since October 2014, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon, Éric Goubault, Sylvie Putot, defended on 23 Nov 2017.
- PhD : Vianney Boeuf, registered at École Polytechnique, since October 2014, thesis supervisor: Stéphane Gaubert, cosupervision: Stéphane Raclot (BSPP), Marianne Akian, Xavier Allamigeon, defended on 18 Dec 2017.
- PhD in progress : Mateusz Skomra, registered at Univ. Paris Saclay since October 2015, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.
- PhD in progress : Jean-Bernard Eytard, registered at Univ. Paris Saclay since October 2015, thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian, Mustapha Bouhtou.
- PhD in progress: Paulin Jacquot, registered at Univ. Paris Saclay since November 2016, thesis supervisor: Stéphane Gaubert, cosupervision: Nadia Oujdane, Olivier Beaude (EDF).
- PhD in progress: Benoît Tran, registered at Univ Paris-Est Marne La Vallée, since September 2017, thesis supervisor: Jean-Philippe Chancelier (ENPC), cosupervision: Marianne Akian.

### 10.2.3. Juries

- X. Allamigeon

- Jury of the PhD thesis of N. Stott (Ecole Polytechnique, examiner, November 2017).
- Jury of the PhD thesis of V. Bœuf (Ecole Polytechnique, examiner, December 2017).
- S. Gaubert
  - Member of hiring committee (Professor position) at Paris 6 University.
  - Jury of the PhD thesis of A. Sagnier (Paris 7, examiner, 2017).
  - Jury of the PhD thesis of G. Loho (TU-Berlin, reviewer, 2017).
  - Jury of the PhD thesis of R. Hess (LAAS, 2017).
  - Jury of the PhD thesis of N. Stott (Ecole Polytechnique, examiner, November 2017).
  - Jury of the PhD thesis of V. Bœuf (Ecole Polytechnique, examiner, December 2017).

### 10.3. Conferences, Seminars

- M. Akian
  - Workshop "Numerical methods for optimal control problems: algorithms, analysis and applications" at INdAM, Roma, Italy, June, 19-23, 2017. Title of the talk: "Probabilistic max-plus schemes for solving Hamilton–Jacobi–Bellman equations".
  - Atelier "Jeux dynamiques à somme nulle: temps discret, temps continu", Fréjus, from 17 to 19 oct 2017. Title of the talk: "Érgodicité des jeux à somme nulle".
- X. Allamigeon
  - "Tropical Mathematics and its Applications" seminar, Warwick University, February 15, 2017. Title of the talk: "Tropicalization of spectrahedra".
  - "Optimisation and Numerical Analysis" seminar, Birmingham University, February 16, 2017. Title of the talk: "Log-barrier interior-point methods are not strongly polynomial".
  - "Computing in Tropical Geometry" workshop, Zuse Institut Berlin, May 11-12, 2017. Title of the talk: "Tropical linear optimization".
  - 8th International Conference on Interactive Theorem Proving, Brasília, September 26-29, 2017. Title of the talk: "A Formalization of Convex Polyhedra based on the Simplex Method".
  - "Computations and Proofs" SpecFun seminar, Saclay, October 16, 2017. Title of the talk: "Log-barrier interior-point methods are not strongly polynomial".
- V. Boeuf
  - INFORMS APS 2017, Applied Probability Society Conference. Chicago, IL, USA, July 10-12, 2017. "An Asymptotic Analysis of Blocking in a Finite Capacity Network with Two Levels."
- J.B. Eytard
  - Congrès annuel de la société française de Recherche Opérationnelle (ROADEF), Feb. 22-24, 2017, Metz. Title of the talk: "Une approche tropicale de la programmation bi-niveau".
  - 15th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WIOPT) , May. 15-17, 2017, Paris. Title of the talk: "A bilevel optimization model for load balancing in mobile networks through price incentives".
  - Séminaire des doctorants du CMAP, Jun. 23, Palaiseau. Title of the talk: "A tropical approach to bilevel programming applied to a price incentives model in mobile data networks".
  - SIAM Applied Algebraic Geometry (SIAMAG), Jul. 31 - Aug. 4, Atlanta. Title of the talk: "A tropical approach to bilevel programming: application to a price incentives model in mobile data networks".

- PGMO Days, Nov. 13-14, 2016, Palaiseau. Title of the talk: “Price incentives in mobile data networks: bilevel programming, competitive equilibria and discrete convexity”.
- S. Gaubert
  - STACS, Hannover, March 2017, The operator approach to entropy games.
  - Noncommutative geometry: number theory, celebration of Alain Connes’ 70th birthday, Shanghai, March 23 - April 7, 2017. Tropical modules, zero-sum games and non-archimedean optimization.
  - Mathematical morphology and its applications to image and signal processing Fontainebleau, Fontainebleau, May 15 – May 17, 2017. Tropical and non-linear Perron-Frobenius methods for optimal control and zero-sum games
  - INDAM Workshop, Roma, June 19–23, 2017. Noncommutative aspects of dynamic programming.
  - Stony Brook Game Theory Conference, July 17-21, 2017. Nonarchimedean convexity and stochastic mean payoff games.
  - OR Berlin, Sep. 5-8, 2017. Tropical spectrahedra and stochastic mean payoff games.
  - Control and Optimization Conference On the occasion of Frédéric Bonnans 60th birthday, EDF Labs Saclay, 15-17 Nov, 2017. Dynamic programming over noncommutative spaces
- D. Jones
  - Tropical Mathematics & its Applications, Birmingham, Nov 15 2017. “A discrete geometry model of fire propagation in urban areas”.
- M. Skomra
  - Séminaire de Géométrie Tropicale, UPMC, Paris, April 26, 2017. Title of the talk: “Tropical spectrahedra”.
  - Computing in Tropical Geometry, Zuse Institute Berlin, May 11–12, 2017. Title of the talk: “Tropical spectrahedra”.
  - SIAM Conference on Optimization, Vancouver, May 22–25, 2017. Title of the talk: “Solving Generic Nonarchimedean Semidefinite Programs using Stochastic Game Algorithms”.
  - International Conference on Effective Methods in Algebraic Geometry (MEGA), Nice, June 12–16, 2017. Title of the talk: “The tropical analogue of the Helton–Nie conjecture is true”.
  - SIAM Conference on Applied Algebraic Geometry, Atlanta, GA, Jul. 31 – Aug. 04, 2017. Title of the talk: “Tropical spectrahedra”.
- C. Walsh
  - Workshop “Compactifications of buildings and symmetric spaces”, Heidelberg, May 16–17, 2017. Title of the talk: “The horofunction compactification of symmetric cones”.
  - Workshop “Order Structures, Jordan Algebras, and Geometry”, Leiden, May 29–June 2, 2017. Title of the talk: “Antitone maps and Euclidean Jordan algebras”.
  - Seminar, Ecole Polytechnique, Palaiseau, June 7, 2017. Title of the talk: “Approximability of convex bodies and volume growth in Hilbert geometries”.
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