Activity Report 2018

Project-Team TROPICAL

Tropical methods: structures, algorithms and interactions

IN COLLABORATION WITH: Centre de Mathématiques Appliquées (CMAP)

RESEARCH CENTER
Saclay - Île-de-France

THEME
Optimization and control of dynamic systems

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Project-Team TROPICAL

Creation of the Team: 2016 January 01, updated into Project-Team: 2018 July 01

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2. Overall Objectives

2.1. Introduction

The project develops tropical methods motivated by applications arising in decision theory (deterministic and stochastic optimal control, game theory, optimization and operations research), in the analysis or control of classes of dynamical systems (including timed discrete event systems and positive systems), in the verification of programs and systems, and in the development of numerical algorithms. Tropical algebra tools are used in interaction with various methods, coming from convex analysis, Hamilton–Jacobi partial differential equations, metric geometry, Perron-Frobenius and nonlinear fixed-point theories, combinatorics or algorithmic complexity. The emphasis of the project is on mathematical modelling and computational aspects.

The subtitle of the *Tropical* project, namely, "structures, algorithms, and interactions", refers to the spirit of our research, including a methodological component, computational aspects, and finally interactions with other scientific fields or real world applications, in particular through mathematical modelling.

2.2. Scientific context

Tropical algebra, geometry, and analysis have enjoyed spectacular development in recent years. Tropical structures initially arose to solve problems in performance evaluation of discrete event systems [62], combinatorial optimization [67], or automata theory [117]. They also arose in mathematical physics and asymptotic analysis [107], [104]. More recently, these structures have appeared in several areas of pure mathematics, in particular in the study of combinatorial aspects of algebraic geometry [96], [131], [120], [101], in algebraic combinatorics [88], and in arithmetics [73]. Also, further applications of tropical methods have appeared, including optimal control [111], program invariant computation [57] and timed systems verification [106], and zero-sum games [2].

The term 'tropical' generally refers to algebraic structures in which the laws originate from optimization processes. The prototypical tropical structure is the max-plus semifield, consisting of the real numbers, equipped with the maximum, thought of as an additive law, and the addition, thought of as a multiplicative law. Tropical objects appear as limits of classical objects along certain deformations ("log-limits sets" of Bergman, "Maslov dequantization", or "Viro deformation"). For this reason, the introduction of tropical tools often yields new insights into old familiar problems, leading either to counterexamples or to new methods and results; see for instance [131], [113]. In some applications, like optimal control, discrete event systems, or static analysis of programs, tropical objects do not appear through a limit procedure, but more directly as a modelling or computation/analysis tool; see for instance [127], [62], [99], [68].

Tropical methods are linked to the fields of positive systems and of metric geometry [115], [11]. Indeed, tropically linear maps are monotone (a.k.a. order-preserving). They are also nonexpansive in certain natural metrics (sup-norm, Hopf oscillation, Hilbert's projective metric, ...). In this way, tropical dynamical systems appear to be special cases of nonexpansive, positive, or monotone dynamical systems, which are studied as part of linear and non-linear Perron-Frobenius theory [105], [3]. Such dynamical systems are of fundamental importance in the study of repeated games [112]. Monotonicity properties are also essential in the understanding of the fixed points problems which determine program invariants by abstract interpretation [75]. The latter problems are actually somehow similar to the ones arising in the study of zero-sum games; see [7]. Moreover, positivity or monotonicity methods are useful in population dynamics, either in a discrete space setting [129] or in a PDE setting [63]. In such cases, solving tropical problems often leads to solutions or combinatorial insights on classical problems involving positivity conditions (e.g., finding equilibria of dynamical systems with nonnegative coordinates, understanding the qualitative and quantitative behavior of growth rates / Floquet eigenvalues [9], etc). Other applications of Perron-Frobenius theory originate from quantum information and control [119], [125].

3. Research Program

3.1. Optimal control and zero-sum games

The dynamic programming approach allows one to analyze one or two-player dynamic decision problems by means of operators, or partial differential equations (Hamilton–Jacobi or Isaacs PDEs), describing the time evolution of the value function, i.e., of the optimal reward of one player, thought of as a function of the initial state and of the horizon. We work especially with problems having long or infinite horizon, modelled by stopping problems, or ergodic problems in which one optimizes a mean payoff per time unit. The determination of optimal strategies reduces to solving nonlinear fixed point equations, which are obtained either directly from discrete models, or after a discretization of a PDE.

The geometry of solutions of optimal control and game problems Basic questions include, especially for stationary or ergodic problems, the understanding of existence and uniqueness conditions for the solutions of dynamic programming equations, for instance in terms of controllability or ergodicity properties, and more generally the understanding of the structure of the full set of solutions of stationary Hamilton–Jacobi PDEs and of the set of optimal strategies. These issues are already challenging in the one-player deterministic case, which is an application of choice of tropical methods, since the Lax-Oleinik semigroup, i.e., the evolution semigroup of the Hamilton-Jacobi PDE, is a linear operator in the tropical sense. Recent progress in the deterministic case has been made by combining dynamical systems and PDE techniques (weak KAM theory [85]), and also using metric geometry ideas (abstract boundaries can be used to represent the sets of solutions [100], [4]). The two player case is challenging, owing to the lack of compactness of the analogue of the Lax-Oleinik semigroup and to a richer geometry. The conditions of solvability of ergodic problems for games (for instance, solvability of ergodic Isaacs PDEs), and the representation of solutions are only understood in special cases, for instance in the finite state space case, through tropical geometry and non-linear Perron-Frobenius methods [47],[14], [3].

Algorithmic aspects: from combinatorial algorithms to the attenuation of the curse of dimensionality Our general goal is to push the limits of solvable models by means of fast algorithms adapted to large scale instances. Such instances arise from discrete problems, in which the state space may so large that it is only accessible through local oracles (for instance, in some web ranking applications, the number of states may be the number of web pages) [86]. They also arise from the discretization of PDEs, in which the number of states grows exponentially with the number of degrees of freedom, according to the "curse of dimensionality". A first line of research is the development of *new approximation methods for the value function*. So far, classical approximations by linear combinations have been used, as well as approximation by suprema of linear or quadratic forms, which have been introduced in the setting of dual dynamic programming and of the so called "max-plus basis methods" [87]. We believe that more concise or more accurate approximations may be obtained by unifying these methods. Also, some max-plus basis methods have been shown to *attenuate the curse of dimensionality* for very special problems (for instance involving switching) [108], [92]. This suggests that the complexity of control or games problems may be measured by more subtle quantities that the mere number of states, for instance, by some forms of metric entropy (for example, certain large scale problems have a low complexity owing to the presence of decomposition properties, "highway hierarchies", etc.). A second line of of our research is the development of *combinatorial algorithms*, to solve large scale zero-sum two-player problems with discrete state space. This is related to current open problems in algorithmic game theory. In particular, the existence of polynomial-time algorithms for games with ergodic payment is an open question. See e.g. [53] for a polynomial time average complexity result derived by tropical methods. The two lines of research are related, as the understanding of the geometry of solutions allows to develop better approximation or combinatorial algorithms.

3.2. Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

Several applications (including population dynamics [9] and discrete event systems [62], [70], [55]) lead to studying classes of dynamical systems with remarkable properties: preserving a cone, preserving an order, or being nonexpansive in a metric. These can be studied by techniques of non-linear Perron-Frobenius theory [3] or metric geometry [10]. Basic issues concern the existence and computation of the "escape rate" (which determines the throughput, the growth rate of the population), the characterizations of stationary regimes (non-linear fixed points), or the study of the dynamical properties (convergence to periodic orbits). Nonexpansive mappings also play a key role in the "operator approach" to zero-sum games, since the one-day operators of games are nonexpansive in several metrics, see [8].

3.3. Tropical algebra and convex geometry

The different applications mentioned in the other sections lead us to develop some basic research on tropical algebraic structures and in convex and discrete geometry, looking at objects or problems with a "piecewise-linear " structure. These include the geometry and algorithmics of tropical convex sets [58], [50], tropical semialgebraic sets [60], the study of semi-modules (analogues of vector spaces when the base field is replaced by a semi-field), the study of systems of equations linear in the tropical sense, investigating for instance the analogues of the notions of rank, the analogue of the eigenproblems [52], and more generally of systems of tropical polynomial equations. Our research also builds on, and concern, classical convex and discrete geometry methods.

3.4. Tropical methods applied to optimization, perturbation theory and matrix analysis

Tropical algebraic objects appear as a deformation of classical objects thought various asymptotic procedures. A familiar example is the rule of asymptotic calculus,

$$e^{-a/\epsilon} + e^{-b/\epsilon} \asymp e^{-\min(a,b)/\epsilon} \ , \qquad e^{-a/\epsilon} \times e^{-b/\epsilon} = e^{-(a+b)/\epsilon} \ , \tag{1}$$

when $\epsilon \to 0^+$. Deformations of this kind have been studied in different contexts: large deviations, zero-temperature limits, Maslov's "dequantization method" [107], non-archimedean valuations, log-limit sets and Viro's patchworking method [132], etc.

This entails a relation between classical algorithmic problems and tropical algorithmic problems, one may first solve the $\epsilon = 0$ case (non-archimedean problem), which is sometimes easier, and then use the information gotten in this way to solve the $\epsilon = 1$ (archimedean) case.

In particular, tropicalization establishes a connection between polynomial systems and piecewise affine systems that are somehow similar to the ones arising in game problems. It allows one to transfer results from the world of combinatorics to "classical" equations solving. We investigate the consequences of this correspondence on complexity and numerical issues. For instance, combinatorial problems can be solved in a robust way. Hence, situations in which the tropicalization is faithful lead to improved algorithms for classical problems. In particular, scalings for the polynomial eigenproblems based on tropical preprocessings have started to be used in matrix analysis [93], [98].

Moreover, the tropical approach has been recently applied to construct examples of linear programs in which the central path has an unexpectedly high total curvature [54], and it has also led to positive polynomial-time average case results concerning the complexity of mean payoff games. Similarly, we are studying semidefinite programming over non-archimedean fields [60], [59], with the goal to better understand complexity issues in classical semidefinite and semi-algebraic programming.

4. Application Domains

4.1. Discrete event systems (manufacturing systems, networks)

One important class of applications of max-plus algebra comes from discrete event dynamical systems [62]. In particular, modelling timed systems subject to synchronization and concurrency phenomena leads to studying dynamical systems that are non-smooth, but which have remarkable structural properties (nonexpansiveness in certain metrics , monotonicity) or combinatorial properties. Algebraic methods allow one to obtain analytical expressions for performance measures (throughput, waiting time, etc). A recent application, to emergency call centers, can be found in [55].

4.2. Optimal control and games

Optimal control and game theory have numerous well established applications fields: mathematical economy and finance, stock optimization, optimization of networks, decision making, etc. In most of these applications, one needs either to derive analytical or qualitative properties of solutions, or design exact or approximation algorithms adapted to large scale problems.

4.3. Operations Research

We develop, or have developed, several aspects of operations research, including the application of stochastic control to optimal pricing, optimal measurement in networks [122]. Applications of tropical methods arise in particular from discrete optimization [68], [69], scheduling problems with and-or constraints [114], or product mix auctions [130].

4.4. Computing program and dynamical systems invariants

A number of programs and systems verification questions, in which safety considerations are involved, reduce to computing invariant subsets of dynamical systems. This approach appears in various guises in computer science, for instance in static analysis of program by abstract interpretation, along the lines of P. and R. Cousot [75], but also in control (eg, computing safety regions by solving Isaacs PDEs). These invariant sets are often sought in some tractable effective class: ellipsoids, polyhedra, parametric classes of polyhedra with a controlled complexity (the so called "templates" introduced by Sankaranarayanan, Sipma and Manna [123]), shadows of sets represented by linear matrix inequalities, disjunctive constraints represented by tropical polyhedra [57], etc. The computation of invariants boils down to solving large scale fixed point problems. The latter are of the same nature as the ones encountered in the theory of zero-sum games, and so, the techniques developed in the previous research directions (especially methods of monotonicity, nonexpansiveness, discretization of PDEs, etc) apply to the present setting, see e.g. [90], [95] for the application of policy iteration type algorithms, or for the application for fixed point problems over the space of quadratic forms [7]. The problem of computation of invariants is indeed a key issue needing the methods of several fields: convex and nonconvex programming, semidefinite programming and symbolic computation (to handle semialgebraic invariants), nonlinear fixed point theory, approximation theory, tropical methods (to handle disjunctions), and formal proof (to certify numerical invariants or inequalities).

5. Highlights of the Year

5.1. Highlights of the Year

- The paper [89] has been included in a list of "10 notable papers published over the last 50 years by the journal Linear Algebra and its applications", at the occasion of the golden anniversary of the journal.
- The article [17] answers an old question in the theory of interior point methods: it provides a counter example showing that log-barrier interior point methods are not strongly polynomial.

6. New Software and Platforms

6.1. Coq-Polyhedra

KEYWORDS: Coq - Polyhedra - Automated theorem proving - Linear optimization
SCIENTIFIC DESCRIPTION: Coq-Polyhedra is a library providing a formalization of convex polyhedra in the Coq proof assistant. While still in active development, it provides an implementation of the simplex method, and already handles the basic properties of polyhedra such as emptiness, boundedness, membership. Several fundamental results in the theory of convex polyhedra, such as Farkas Lemma, duality theorem of linear programming, and Minkowski Theorem, are also formally proved.

The formalization is based on the Mathematical Components library, and makes an extensive use of the boolean reflection methodology.
FUNCTIONAL DESCRIPTION: Coq-Polyhedra is a library which aims at formalizing convex polyhedra in Coq

- Participants: Xavier Allamigeon, Vasileios Charisopoulos and Ricardo Katz
- Partner: CIFASIS
- Contact: Xavier Allamigeon
- Publications: A Formalization of Convex Polyhedra Based on the Simplex Method - A Formalization of Convex Polyhedra Based on the Simplex Method - First steps in the formalization of convex polyhedra in Coq
- URL: https://github.com/nhojem/Coq-Polyhedra

7. New Results

7.1. Optimal control and zero-sum games

7.1.1. Fixed points of order preserving homogeneous maps and zero-sum games

Participants: Marianne Akian, Stéphane Gaubert.

In a series of joint works with Antoine Hochart, we apply methods of non-linear fixed point theory to zero-sum games.

A key issue is the solvability of the ergodic equation associated to a zero-sum game with finite state space, i.e., given a dynamic programming operator T associated to an undiscounted problem, one looks for a vector u, called the bias, and for a scalar λ, the ergodic constant, such that $T(u) = \lambda e + u$. The bias vector is of interest as it allows to determine optimal stationnary strategies.

In [14], we studied zero-sum games with perfect information and finite action spaces, and showed that the set of payments for which the bias vector is not unique (up to an additive constant) coincides with the union of lower dimensional cells of a polyhedral complex, in particular, the bias vector is unique, generically. We provided an application to perturbation schemes in policy iteration.

In [36], we apply game theory methods to the study of the nonlinear eigenproblem for homogeneous order preserving self maps of the interior of the cone. We show that the existence and uniqueness of an eigenvector is governed by combinatorial conditions, involving dominions (sets of states "controlled" by one of the two players). In this way, we characterize the situation in which the existence of an eigenvector holds independently of perturbations, and we solve an open problem raised in [91].

In [15], we provide a representation theorem for "payment free" Shapley operators, showing that these are characterized by monotonicity and homogeneity axioms [15]. This extends to the two-player case known representation theorems for risk measures.

7.1.2. Nonlinear fixed point methods to compute joint spectral raddi of nonnegative matrices

Participants: Stéphane Gaubert, Nikolas Stott.

In [29], we introduce a non-linear fixed point method to approximate the joint spectral radius of a finite set of nonnegative matrices. We show in particular that the joint spectral radius is the limit of the eigenvalues of a family of non-linear risk-sensitive type dynamic programming operators. We develop a projective version of Krasnoselskii-Mann iteration to solve these eigenproblems, and report experimental results on large scale instances (several matrices in dimensions of order 1000 within a minute). The situation in which the matrices are not nonnegative is amenable to a similar approach [94].

7.1.3. Probabilistic and max-plus approximation of Hamilton-Jacobi-Bellman equations

Participants: Marianne Akian, Eric Fodjo.

The PhD thesis of Eric Fodjo concerns stochastic control problems obtained in particular in the modelisation of portfolio selection with transaction costs. The dynamic programming method leads to a Hamilton-Jacobi-Bellman partial differential equation, on a space with a dimension at least equal to the number of risky assets. The curse of dimensionality does not allow one to solve numerically these equations for a large dimension (greater to 5). We propose to tackle these problems with numerical methods combining policy iterations, probabilistic discretisations, max-plus discretisations, in order to increase the possible dimension.

We consider fully nonlinear Hamilton-Jacobi-Bellman equations associated to diffusion control problems with finite horizon involving a finite set-valued (or switching) control and possibly a continuum-valued control. In [46], we constructed a lower complexity probabilistic numerical algorithm by combining the idempotent expansion properties obtained by McEneaney, Kaise and Han [103], [109] for solving such problems with a numerical probabilistic method such as the one proposed by Fahim, Touzi and Warin [82] for solving some fully nonlinear parabolic partial differential equations, when the volatility does not oscillate too much. In [32], [33], we improve the method of Fahim, Touzi and Warin by introducing probabilistic schemes which are monotone without any restrictive condition, allowing one to solve fully nonlinear parabolic partial differential equations with general volatilities. We study the convergence and obtain error estimates when the parameters and the value function are bounded. The more general quadratic growth case has been studied in the PhD manuscript [12].

7.1.4. Tropical-SDDP algorithms for stochastic control problems involving a switching control

Participants: Marianne Akian, Duy Nghi Benoît Tran.

The PhD thesis of Benoît Tran, supervised by Jean-Philippe Chancelier (ENPC) and Marianne Akian concerns the numerical solution of the dynamic programming equation of discrete time stochastic control problems.

Several methods have been proposed in the litterature to bypass the curse of dimensionality difficulty of such an equation, by assuming a certain structure of the problem. Examples are the max-plus based method of McEneaney [110], [111], the stochastic max-plus scheme proposed by Zheng Qu [118], the stochastic dual dynamic programming (SDDP) algorithm of Pereira and Pinto [116], the mixed integer dynamic approximation scheme of Philpott, Faisal and Bonnans [61], the probabilistic numerical method of Fahim, Touzi and Warin [82]. We propose to associate and compare these methods in order to solve more general structures.

In a first work [35], we build a common framework for both the SDDP and a discrete time and finite horizon version of Zheng Qu's algorithm for deterministic problems involving a finite set-valued (or switching) control and a continuum-valued control. We propose an algorithm that generates monotone approximations of the value function as a pointwise supremum, or infimum, of basic (affine or quadratic for example) functions which are randomly selected. We give sufficient conditions that ensure almost sure convergence of the approximations to the value function.

7.1.5. Parametrized complexity of optimal control and zero-sum game problems

Participants: Marianne Akian, Stéphane Gaubert, Omar Saadi.

As already said above, the dynamic programing approach to optimal control and zero-sum game problems suffers of the curse of dimensionality. The aim of the PhD thesis is to unify different techniques to bypass this difficulty, in order to obtain new algorithms and new complexity results.

As a first step, we worked to extend an algorithm proposed by Sidford et al. in [126]. There, they proposed a randomized value iteration algorithm which improves the usual complexity bounds of the value iteration for *discounted* Markov Decision Problems (discrete time stochastic control problems). In a joint work with Zheng Qu (Hong Kong University), we are extending this algorithm to the ergodic (mean payoff) case, exploiting techniques from non-linear spectral theory [48]; this extension covers as well the case of two players (zero-sum).

7.2. Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

7.2.1. Order isomorphisms and antimorphisms on cones

Participant: Cormac Walsh.

We have been studying non-linear operators on open cones, particularly ones that preserve or reverse the order structure associated to the cone. A bijective map that preserves the order in both directions is called an order isomorphism. Those that reverse the order in both directions are order antimorphisms. These are closely related to the isometries of the Hilbert and Thompson metrics on the cone.

Previously, we have shown [133] that if there exists an antimorphism on a finite-dimensional open cone that is homogeneous of degree -1, then the cone must be a symmetric cone, that is, have a transitive group of linear automorphisms and be self-dual. This result was improved in [44], where we showed that the homogeneity assumption is not actually necessary: every antimorphism on a cone is automatically homogeneous of degree -1.

The study of the order isomorphisms of a cone goes back to Alexandrov and Zeeman, who considered maps preserving the light cone that arises in special relativity. This work was extended to more general cones by Rothaus; Noll and Schäffer; and Artstein-Avidan and Slomka. It was shown, in the finite-dimensional case, that all isomorphisms are linear if the cone has no one-dimensional factors. There are also some results in infinite dimension—however these are unsatisfactory because of the strong assumptions that must be made in order to get the finite-dimensional techniques to work. For example, a typical assumption is that the cone is the convex hull of its extreme rays, which is overly restrictive in infinite dimension. Using different techniques more suited to infinite dimension, we have been developing a necessary and sufficient criterion on the geometry of a cone for all its isomorphisms to be linear.

7.2.2. Horofunction compactifications of symmetric spaces

Participant: Cormac Walsh.

This work is in collaboration with Thomas Haettel (Montpellier), Anna-Sofie Schilling (Heidelberg), Anna Wienhard (Heidelberg).

The symmetric spaces form a fascinating class of geometrical space. These are the spaces in which there is a point reflection through every point. An example is the space $\mathrm{Pos}(\mathbb{C}, n)$ of positive definite $n \times n$ Hermitian matrices.

The interesting metrics on such spaces are the ones that are invariant under all the symmetries, in particular the invariant Finsler metrics. When the symmetric space is non-compact, as in the example just referred to, it is profitable to study the horofunction boundary of such metrics.

An important technique in trying to understand symmetric spaces is to look at their *flats*. These are subspaces that are, as their name suggests, flat in some sense. Because of the abundance of symmetries, there are many flats; indeed, every pair of points lies in a flat. Furthermore, given any two flats, there is a symmetry taking one to the other, and so they are all alike. It turns out that the restriction of an invariant Finsler metric to a single flat determines the metric everywhere, and gives the flat the geometry of a normed space.

Symmetric spaces can be compactified by means of the Satake compactification. In fact, there are several such compactifications, one associated to each irreducible faithful representation of the invariance group of the space. In [41], we show that each Satake compactification can be constructed as a horofunction compactification by choosing an appropriate invariant Finsler metric. In fact, the metrics we construct have polyhedral balls on the flat.

An important step in the proof is to show that the closure of a flat in the horofunction compactification of the symmetric space is the same as the horofunction compactification of the flat viewed as a metric space in its own right. This is not true for every metric space, since in general one might not be able to distinguish horofunctions by looking at a subspace.

7.2.3. The set of minimal upper bounds of two matrices in the Loewner order

Participant: Nikolas Stott.

A classical theorem of Kadison shows that the space of symmetric matrices equipped with the Loewner order is an anti-lattice, meaning that two matrices have a least upper bound if and only if they are comparable. In [24], we refined this theorem by characterizing the set of minimal upper bounds: we showed that it is homeomorphic to the quotient space $O(p) \setminus O(p,q)/O(q)$, where $O(p,q)$ denotes the orthogonal group associated to the quadratic form with signature (p,q), and $O(p)$ denotes the standard pth orthogonal group.

7.2.4. Generalization of the Hellinger distance

Participant: Stéphane Gaubert.

In [64] (joint work with Rajendra Bhatia of Ashoka University and Tanvi Jain, Indian Statistic Institute, New Delhi), we study some generalizations of the Hellinger distance to the space of positive definite matrices.

7.2.5. Spectral inequalities for nonnegative tensors and their tropical analogues

Participant: Stéphane Gaubert.

In [39] (joint work with Shmuel Friedland, University of Illinois at Chicago) we extend some characterizations and inequalities for the eigenvalues of nonnegative matrices, such as Donsker-Varadhan, Friedland-Karlin, Karlin-Ost inequalities, to nonnegative tensors. These inequalities are related to a correspondence between nonnegative tensors and ergodic control: the logarithm of the spectral radius of a tensor is given by the value of an ergodic problem in which instantaneous payments are given by a relative entropy. Some of these inequalities involve the tropical spectral radius, a limit of the spectral radius which we characterize combinatorially as the value of an ergodic Markov decision process.

7.3. Tropical algebra and convex geometry

7.3.1. Formalizing convex polyhedra in Coq

Participants: Xavier Allamigeon, Ricardo Katz [Conicet, Argentine].

In [20], we have made the first steps of a formalization of the theory of convex polyhedra in the proof assistant Coq. The originality of our approach lies in the fact that our formalization is carried out in an effective way, in the sense that the basic predicates over polyhedra (emptiness, boundedness, membership, etc) are defined by means of Coq programs. All these predicates are then proven to correspond to the usual logical statements. The latter take the form of the existence of certificates: for instance, the emptiness of a polyhedron is shown to be equivalent to the existence of a certificate *a la* Farkas. This equivalence between Boolean predicates and formulas living in the kind `Prop` is implemented by using the boolean reflection methodology, and the supporting tools provided by the Mathematical Components library and its tactic language. The benefit of the effective nature of our approach is demonstrated by the fact that we easily arrive at the proof of important results on polyhedra, such as several versions of Farkas Lemma, duality theorem of linear programming, separation from convex hulls, Minkowski Theorem, etc.

Our effective approach is made possible by implementing the simplex method inside Coq, and proving its correctness and termination. Two difficulties need to be overcome to formalize it. On the one hand, we need to deal with its termination. More precisely, the simplex method iterates over the so-called bases. Its termination depends on the specification of a pivoting rule, whose aim is to determine, at each iteration, the next basis. In this work, we have focused on proving that the lexicographic rule ensures termination. On the other hand, the simplex method is actually composed of two parts. The part that we previously described, called Phase II, requires an initial basis to start with. Finding such a basis is the purpose of Phase I. It consists in building an extended problem (having a trivial initial basis), and applying to it Phase II. Both phases need to be formalized to obtain a fully functional algorithm.

7.3.2. Tropical totally positive matrices

Participant: Stéphane Gaubert.

In [22] (joint work with Adi Niv) we investigate the tropical analogues of totally positive and totally non-negative matrices, i.e, the images by the valuation of the corresponding classes of matrices over a non-archimedean field. We show in particular that tropical totally positive matrices essentially coincide with the Monge matrices (defined by the positivity of 2×2 tropical minors), arising in optimal transport, and compare the set of tropical totally positive matrices with the tropicalization of the totally positive Grassmannian.

7.3.3. Tropical compound matrix identities

Participants: Marianne Akian, Stéphane Gaubert.

A number of polynomial identities in tropical semirings can be derived from their classical analogues by application of a transfer principle [49], [51]. In [16], joint with Adi Niv, we prove identities on compound matrices in extended tropical semirings, which cannot be obtained by transfer principles, but are rather obtained by combinatorial methods. Such identities include analogues to properties of conjugate matrices, powers of matrices and $\mathrm{adj}\,(A)\,\det\,(A)^{-1}$, all of which have implications on the eigenvalues of the corresponding matrices. A tropical Sylvester-Franke identity is provided as well.

7.3.4. Group algebra in characteristic one and invariant distances over finite groups

Participant: Stéphane Gaubert.

In [21] (joint work with Dominique Castella), we investigated a tropical analogue of group algebras. We studied tropical characters and related them to invariant distances over groups.

7.3.5. Volume and integer points of tropical polytopes

Participant: Stéphane Gaubert.

We investigate in [40] (joint work with Marie McCaig) the volume of tropical polytopes, as well as the number of integer points contained in integer polytopes. We proved that even approximating these values for a tropical polytope given by its vertices is hard, with no approximation algorithm with factor $2^{\mathrm{poly}(m,n)}$ existing unless P = NP.

7.4. Tropical methods applied to optimization, perturbation theory and matrix analysis

7.4.1. Tropicalization of the central path and application to the complexity of interior point methods

Participants: Xavier Allamigeon, Stéphane Gaubert.

This work is in collaboration with Pascal Benchimol (EDF Labs) and Michael Joswig (TU Berlin).

In optimization, path-following interior point methods are driven to an optimal solution along a trajectory called the central path. The *central path* of a linear program $\mathrm{LP}(A, b, c) \equiv \min\{c \cdot x \mid Ax \leq b,\ x \geq 0\}$ is defined as the set of the optimal solutions (x^μ, w^μ) of the barrier problems:

$$\begin{array}{ll} \text{minimize} & c \cdot x - \mu(\sum_{j=1}^{n} \log x_j + \sum_{i=1}^{m} \log w_i) \\ \text{subject to} & Ax + w = b,\ x > 0,\ w > 0 \end{array}$$

While the complexity of interior point methods is known to be polynomial, an important question is to study the number of iterations which are performed by interior point methods, in particular whether it can be bounded by a polynomial in the dimension (mn) of the problem. This is motivated by Smale 9th problem [128], on the existence of a strongly polynomial complexity algorithm for linear programming. So far, this question has been essentially addressed though the study of the curvature of the central path, which measures how far a path differs from a straight line, see [77], [76], [79], [78]. In particular, by analogy with the classical Hirsch conjecture, Deza, Terlaky and Zinchencko [78] proposed the "continuous analogue of the Hirsch conjecture", which says that the total curvature of the central path is linearly bounded in the number m of constraints.

In a work of X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig [17], we prove that primal-dual log-barrier interior point methods are not strongly polynomial, by constructing a family of linear programs with $3r + 1$ inequalities in dimension $2r$ for which the number of iterations performed is in $\Omega(2^r)$. The total curvature of the central path of these linear programs is also exponential in r, disproving the continuous analogue of the Hirsch conjecture.

Our method is to tropicalize the central path in linear programming. The tropical central path is the piecewise-linear limit of the central paths of parameterized families of classical linear programs viewed through logarithmic glasses. We give an explicit geometric characterization of the tropical central path, as a tropical analogue of the barycenter of a sublevel set of the feasible set induced by the duality gap. We study the convergence properties of the classical central path to the tropical one. This allows us to show that that the number of iterations performed by interior point methods is bounded from below by the number of tropical segments constituting the tropical central path.

7.4.2. Tropicalization of semidefinite programming and its relation with stochastic games

Participants: Xavier Allamigeon, Stéphane Gaubert, Mateusz Skomra.

Semidefinite programming consists in optimizing a linear function over a spectrahedron. The latter is a subset of $\mathbb{R}^n$ defined by linear matrix inequalities, i.e., a set of the form

$$\left\{x \in \mathbb{R}^n : Q^{(0)} + x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \succeq 0\right\}$$

where the $Q^{(k)}$ are symmetric matrices of order m, and $\succeq$ denotes the Loewner order on the space of symmetric matrices. By definition, $X \succeq Y$ if and only if $X - Y$ is positive semidefinite.

Semidefinite programming is a fundamental tool in convex optimization. It is used to solve various applications from engineering sciences, and also to obtain approximate solutions or bounds for hard problems arising in combinatorial optimization and semialgebraic optimization.

A general issue in computational optimization is to develop combinatorial algorithms for semidefinite programming. Indeed, semidefinite programs are usually solved via interior point methods. However, the latter provide an approximate solution in a polynomial number of iterations, provided that a strictly feasible initial solution. Semidefinite programming becomes a much harder matter if one requires an exact solution. The feasibility problem belongs to $\mathsf{NP}_{\mathbb{R}} \cap \mathsf{coNP}_{\mathbb{R}}$, where the subscript $\mathbb{R}$ refers to the BSS model of computation. It is not known to be in NP in the bit model.

We address semidefinite programming in the case where the field $\mathbb{R}$ is replaced by a nonarchimedean field, like the field of Puiseux series. In this case, methods from tropical geometry can be applied and are expected to allow one, in generic situations, to reduce semialgebraic problems to combinatorial problems, involving only the nonarchimedean valuations (leading exponents) of the coefficients of the input.

To this purpose, we first study tropical spectrahedra, which are defined as the images by the valuation of nonarchimedean spectrahedra. We establish that they are closed semilinear sets, and that, under a genericity condition, they are described by explicit inequalities expressing the nonnegativity of tropical minors of order 1 and 2. These results are presented in the preprint [60], with further results in the PhD thesis [13].

We show in [18] that the feasibility problem for a generic tropical spectrahedron is equivalent to solving a stochastic mean payoff game (with perfect information). The complexity of these games is a long-standing open problem. They are not known to be polynomial, however they belong to the class $\mathsf{NP} \cap \mathsf{coNP}$, and they can be solved efficiently in practice. This allows to apply stochastic game algorithms to solve nonarchimedean semidefinite feasibility problems. We obtain in this way both theoretical bounds and a practicable method which solves some large scale instances.

A long-standing problem is to characterize the convex semialgebraic sets that are SDP representable, meaning that they can be represented as the image of a spectrahedron by a (linear) projector. Helton and Nie conjectured that every convex semialgebraic set over the field of real numbers are SDP representable. Recently, [124] disproved this conjecture. In [19], we show, however, that the following result, which may be thought of as a tropical analogue of this conjecture, is true: over a real closed nonarchimedean field of Puiseux series, the convex semialgebraic sets and the projections of spectrahedra have precisely the same images by the nonarchimedean valuation. The proof relies on game theory methods and on our previous results [60] and [18].

In [27] and [13], we exploit the tropical geometry approach to introduce a condition number for stochastic mean payoff games (with perfect information). This condition number is defined as the maximal radius of a ball in Hilbert's projective metric, contained in a primal or dual feasible set. We show that the convergence time of value iteration is governed by this condition number, and derive fixed parameter tractability results.

7.4.3. Tropical polynomial systems and colorful interior of convex bodies

Participants: Marianne Akian, Marin Boyet, Xavier Allamigeon, Stéphane Gaubert.

The starting PhD thesis work of Marin Boyet, deals with the solution of tropical polynomial systems, with motivations from call center performance evaluation (see Section 7.6.1). We introduced a notion of colorful interior of a family of convex bodies, and showed that the solution of such a polynomial system reduces to linear programming if one knows a vector in the colorful interior of an associated family of Newton polytopes. Further properties of colorful interiors are currently investigated.

7.5. Tropical algebra, number theory and directed algebraic topology

7.5.1. An arithmetic site of Connes-Consani type for number fields with narrow class number 1

Participant: Aurélien Sagnier.

In 1995, A. Connes ([71]) gave a spectral interpretation of the zeroes of the Riemann zeta function involving the action of $\mathbb{R}_+^*$ on the sector $X = \mathbb{Q}_+^\times \smallsetminus \mathbb{A}_\mathbb{Q}/\widehat{\mathbb{Z}}^\times$ of the adele class space $\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*$ of the field of rational numbers. In [72], [74], the action of $\mathbb{R}_+^*$ on this sector X was shown to have a natural interpretation in algebraic geometry. This interpretation requires the use of topos theory as well as of the key ingredient of characteristic one namely the semifield $\mathbb{R}_{\max}$ familiar in tropical geometry. The automorphism group of this semifield is naturally isomorphic to $\mathbb{R}_+^*$ and plays the role of the Frobenius. As it turns out, its action on the points of a natural semiringed topos corresponds canonically to the above action on X. This semiringed topos is called the arithmetic site. In my PhD, I extended the construction of the arithmetic site, replacing the field of rational numbers by certain number fields. I considered the simplest complex case, namely that of imaginary quadratic fields on which we assume that the units are not reduced to ± 1 that is when K is either $\mathbb{Q}(\imath)$ or $\mathbb{Q}(\imath\sqrt{3})$. These results are presented in the submitted article [121]. In a further work, developed this year, I extended this construction, dealing now with number fields K with narrow class number 1. In fact, if we denote $\mathcal{U}_{\mathcal{O}_K}^+$ the totally positive units of $\mathcal{O}_K$, $\mathcal{O}_K^+$ the totally positive integers, $\widehat{\mathcal{O}_K^+}$ the topos of sets with a multiplicative action of totally positive integers of $\mathcal{O}_K^+$, $D_K = \text{Conv}(\mathcal{U}_{\mathcal{O}_K}^+)$ and $\mathcal{C}_{\mathcal{O}_K} = \text{Semiring}(\{\lambda D_K/\lambda \in \mathcal{O}_K^+\}) \cup \{\varnothing, \{0\}\}$, we consider the semiringed topos $(\widehat{\mathcal{O}_K^+}, (\mathcal{C}_{\mathcal{O}_K}, \text{Conv}(\bullet \cup \bullet), +))$ and show for it similar properties as the one shown in my PhD thesis for the arithmetic sites associated to imaginary quadratic fields with class number 1 by adapting to this case the technics used in my PhD thesis, Shintani units theorem, and some remarks A.Connes made on my PhD which appear as an appendix of the article [121]. Here again tropical algebra play a crucial role in the geometrical constructions.

7.5.2. *Tropical tensor products*

Participants: Stéphane Gaubert, Aurélien Sagnier.

Tensors products of modules over semifields of characteristic one, like the Boolean or tropical semifields, have appeared recently, with motivations from arithmetics, in work by Connes and Consani, towards an intersection theory for arithmetic and scaling sites (spaces they have built and which are closely related to the zeroes of the Riemann zeta function). Algebraic and topological tropical tensors products were constructed in a different way by Litvinov and collaborators: here, tropical tensors are sums of "rank one" expressions, similar to the ones used in the approximation of large data sets or of functions of many parameters. We show that the canonical notion of tropical tensor product, defined in terms of the usual universal problem, differs from the definition arising from approximation theory, but that the latter can be recovered from the former by a certain "reduction" operation. We illustrate these results by computing several basic examples of categorical tensors products, including spaces of convex sets and functions.

7.5.3. *Directed topological complexity and control*

Participant: Aurélien Sagnier.

This is a joint work with Michael Farber and Eric Goubault.

The view we are taking here is that of topological complexity, as defined in [83], adapted to directed topological spaces.

Let us briefly motivate the interest of a directed topological complexity notion. It has been observed that the very important planification problem in robotics boils down to, mathematically speaking, finding a section to the path space fibration $\chi : PX = X^I \to X \times X$ with $\chi(p) = (p(0), p(1))$. If this section is continuous, then the complexity is the lowest possible (equal to one), otherwise, the minimal number of discontinuities that would encode such a section would be what is called the topological complexity of X. This topological complexity is both understandable algorithmically, and topologically, e.g. as s having a continuous section is equivalent to X being contractible. More generally speaking, the topological complexity is defined as the Schwartz genus of the path space fibration, i.e. is the minimal cardinal of partitions of $X \times X$ into “nice” subspaces F_i such that $s_{F_i} : F_i \to PX$ is continuous.

This definition perfectly fits the planification problem in robotics where there are no constraints on the actual control that can be applied to the physical apparatus that is supposed to be moved from point a to point b. In many applications, a physical apparatus may have dynamics that can be described as an ordinary differential equation in the state variables $x \in \mathbb{R}^n$ and in time t, parameterized by control parameters $u \in \mathbb{R}^p$, $\dot{x}(t) = f(t, x(t))$. These parameters are generally bounded within some set U, and, not knowing the precise control law (i.e. parameters u as a function of time t) to be applied, the way the controlled system can evolve is as one of the solutions of the differential inclusion $\dot{x}(t) \in F(t, x(t))$ where $F(t, x(t))$ is the set of all $f(t, x(t), u)$ with $u \in U$. Under some classical conditions, this differential inclusion can be proven to have solutions on at least a small interval of time, but we will not discuss this further here. Under the same conditions, the set of solutions of this differential inclusion naturally generates a dspace (a very general structure of directed space, where a preferred subset of paths is singled out, called directed paths, see e.g. [97]). Now, the planification problem in the presence of control constraints equates to finding sections to the analogues to the path space fibration (That would most probably not qualify for being called a fibration in the directed setting) taking a dipath to its end points. This notion is developed in this article, and we introduce a notion of directed homotopy equivalence that has precisely, and in a certain non technical sense, minimally, the right properties with respect to this directed version of topological complexity.

This notion of directed topological complexity also has applications in informatics where a directed space can be used to model the space of all possible executions of a concurrent process (ie when several running programs must share common limited ressources).

In a recent prepublication [84], after defining the notion of directed topological complexity,this invariant (directed topological complexity) is studied for directed spheres and directed graphs.

7.6. Applications

7.6.1. Performance evaluation of an emergency call center

Participants: Xavier Allamigeon, Stéphane Gaubert.

Since 2014, we have been collaborating with Préfecture de Police (Régis Reboul and LcL Stéphane Raclot), more specifically with Brigade de Sapeurs de Pompiers de Paris (BSPP) and Direction de Sécurité de Proximité de l'agglomération parisienne (DSPAP), on the performance evaluation of the new organization to handle emergency calls to firemen and policemen in the Paris area. We developed analytical models, based on Petri nets with priorities, and fluid limits, see [55], [56], [65]. In 2018, we performed specific case studies, with several students of École polytechnique: Laetitia de Coudenhove, Julie Poulet, Céline Moucer and Julia Escribe.

7.6.2. Tropical models of fire propagation in urban areas

Participants: Stéphane Gaubert, Daniel Jones.

As part of the team work in the ANR project Democrite, we developed a model of fire propagation in urban areas, involving a deterministic analogue of first passage percolation. We showed that the fire tends to propagate according to a polyhedral shape, and derived metric limit theorems, exploiting discrete convexity results à la Shapley-Folkman. We validated this approach by simulations, on the fire following Kobe earthquake in 1995. The polyhedral shape is also apparent in historical fires, like the great fire of London (1666). These results are announced in [28].

7.6.3. Smart Data Pricing

Participants: Marianne Akian, Jean-Bernard Eytard, Stéphane Gaubert.

This work is in collaboration with Mustapha Bouhtou (Orange Labs) and with Gleb Koshevoy (Russian academy of Science).

The PhD work [81] of Jean-Bernard Eytard concerns the optimal pricing of data trafic in mobile networks. We developed a bilevel programming approach, allowing to an operator to balance the load in the network through price incentives. We showed that a subclass of bilevel programs can be solved in polynomial time, by combining methods of tropical geometry and of discrete convexity. This work is presented in [80] and also in [34]. In a followup work, presented in [81], we managed to extend these results to wider classes of bilevel problems, and to relate them to competitive equilibria problems.

7.6.4. Game theory models of decentralized mechanisms of pricing of the smart grid

Participants: Stéphane Gaubert, Paulin Jacquot.

This work is in collaboration with Nadia Oudjane, Olivier Beaude and Cheng Wan (EDF Labs).

The PhD work of Paulin Jacquot concerns the application of game theory techniques to pricing of energy. We are developing a game theory framework for demand side management in the smart grid, in which users have movable demands (like charging an electric vehicle). We compared in particular the daily and hourly billing mechanisms. The latter, albeit more complex to analyse, has a merit as it incitates the user to move his or her consumption at off peak hours. We showed the Nash equilibrium is unique, under some assumptions, and gave theoretical bounds of the price of anarchy of the game with a hourly billing, showing this mechanism remains efficient while being more “fair” than the daily billing. We proposed and tested decentralized algorithms to compute the Nash equilibrium. These contributions are presented in [102], [23].

Another work, by Paulin Jacquot and Cheng Wan, deals with limit theorems for atomic games with a large number of players [30], [42].

8. Bilateral Contracts and Grants with Industry

8.1. Bilateral Contracts with Industry

- Yield management methods applied to the pricing of data trafic in mobile networks. CRE (research contract) with Orange Labs (Orange Labs partner: Mustapha Bouhtou).
- Decentralized mechanisms of operation of power systems: equilibria and efficiency. Collaboration with Nadia Oudjane and Olivier Beaude from EDF-labs, with the PhD work of Paulin Jacquot (CIFRE PhD), supervised by Stéphane Gaubert.
- Stochastic optimization of multiple flexibilities and energies in micro-grids, collaboration with Wim Van Ackooij, from EDF labs, with the PhD work of Maxime Grangereau (CIFRE PhD), supervised by Emmanuel Gobet (CMAP) and cosupervised by Stéphane Gaubert.

9. Partnerships and Cooperations

9.1. National Initiatives

9.1.1. ANR

- Projet ANR MALTHY (Méthodes ALgébriques pour la vérification de modèles Temporisés et HYbrides), responsable T. Dang. Partenaires : Verimag, CEA LIST, Inria Rennes, Inria Saclay, VISEO/Object Direct.
- Projet ANR DEMOCRITE ("DEmonstrateur d’un MOteur de Couverture des Risques sur un TErritoire), responsable Emmanuel Lapébie (CEA). Partenaires : CEA-GRAMAT, BSPP, Inria Saclay (Maxplus), Institut PPRIME - UPR3346 (CNRS, Univ. Poitiers, ISAE-ENSMA), IPSIS, SYSTEL, ARMINES-E.M. Alès-ISR, CERDACC (Univ. de Haute-Alsace).

- Projet ANR JCJC CAPPS ("Combinatorial Analysis of Polytopes and Polyhedral Subdivisions"), responsable Arnau Padrol (IMJ-PRG, Sorbonne Université). Partenaires : IMJ-PRG (Sorbonne Université), Inria Saclay (Tropical), LIGM (Université Paris-Est Marne-la-Vallée), LIF (Université Aix-Marseille), CERMICS (École Nationale des Ponts et Chaussées), LIX (École Polytechnique).

9.1.2. Programme Gaspard Monge pour l'Optimisation

- Projet intitulé "Méthodes tropicales pour l'optimisation", responsable X. Allamigeon, faisant intervenir M. Akian, V. Boeuf, S. Gaubert, A.Hochart, R. Katz, et M. Skomra.

9.2. European Initiatives

9.2.1. Collaborations with Major European Organizations

- Partner: Michael Joswig, TU-Berlin.
- Topic : Tropical geometry.

9.3. International Initiatives

9.3.1. Inria International Partners

9.3.1.1. Informal International Partners

- Collaboration with Ricardo D. Katz, CIFASIS-CONICET, Rosario (Argentina). Research invitation at CMAP during 2 months.
- Collaboration with Shmuel Friedland, University of Illinois at Chicago (invitation or Stéphane Gaubert at Chicago of one week in May 2018).
- Collaboration with Alejandro Jofre, CMM, University of Chili, Santiago: invitation of Paulin Jacquot of two months (May-June) 2018.
- Collaboration with Zheng Qu, Math. Department, Hongk Kong University.

9.3.2. Participation in Other International Programs

- Collaboration with Gleb Koshevoy, Poncelet Laboratory, Moscow (research invitation of Gleb Koshevoy at CMAP during one week).
- Collaboration with Aris Daniilidis, from CMM, University of Chile, Santiago.

9.4. International Research Visitors

9.4.1. Visits of International Scientists

- Aris Daniilidis, from CMM, University of Chile, Santiago, Sept-Dec 2018. Invited to École polytechnique within the Gaspard Monge Visiting Professor Program (GMVP) of École polytechnique, with the support of Fondation de l'École polytechnique.
- Roberto Bobadilla Solari, invited PhD student, University of Chile, Santiago, from Sep 2018 until Nov 2018, funded by Inria, associated to the visit of Aris Daniilidis.
- Gonzalo Flores Garcia, invited PhD student, University of Chile, Santiago, from Sep 2018 until Dec 2018, associated to the visit of Aris Daniilidis.
- Sebastian Tapia Garcia, invited PhD student, University of Chile, Santiago, funded by GMVP, associated to the visit of Aris Daniilidis.
- Francisco Javier Antonio Venegas Martinez, invited PhD student, University of Chile, Santiago, from Sep 2018 until Nov 2018, funded by Inria, associated to the visit of Aris Daniilidis.
- Rajendra Bhatia (Ashoka University, India), June 2018.
- Gleb Koshevoy (Russian Academy of Siences), Nov 2018.

9.4.1.1. Internships

- Raphael Pellegrin (Imperial College, London), research summer internship, on tropical positivstellensätze.
- Marin Boyet (École Nationale Supérieure des Mines de Paris), research internship, colorful interior of convex bodies and solvability of tropical polynomial systems.

9.4.2. Visits to International Teams

9.4.2.1. Research Stays Abroad

- Marianne Akian, Institut Mittag Leffler, Jan 15-March 3, 2018.
- Stéphane Gaubert, Institut Mittag Leffler, Jan 15-March 3, 2018.
- Stéphane Gaubert, joint invitation by the Statistics Department of University of Chicago (Lek Heng Lim) and the Maths and Computer Science Department of University of Illinois at Chicago (Shmuel Friedland), May 20-26, 2018.

10. Dissemination

10.1. Promoting Scientific Activities

10.1.1. Scientific Events Organisation

10.1.1.1. General Chair, Scientific Chair

- Stéphane Gaubert is the coordinator of the Gaspard Monge Program for Optimization, Operations Research and their interactions with data sciences (PGMO), a corporate sponsorhip program, operated by Fondation Mathématique Jacques Hadamard, supported by Criteo, EDF, Orange and Thales, see http://www.fondation-hadamard.fr/fr/pgmo/.

10.1.1.2. Member of the Organizing Committees

- S. Gaubert co-organizes the "Séminaire Parisien d'Optimisation" at Institut Henri Poincaré. https://sites.google.com/site/spoihp/.

10.1.2. Scientific Events Selection

10.1.2.1. Chair of Conference Program Committees

- S. Gaubert, Chair of the PGMO days, EDF Labs Paris Saclay, Nov 20-21, 2018. http://www.fondation-hadamard.fr/fr/pgmo/pgmodays
- S. Gaubert, coorganizer, with A. Daniilidis and S. Tapia, of the workshop "Dynamical Aspects in Variational Analysis", École polytechnique, Dec. 14th, 2018. http://www.cmap.polytechnique.fr/~gaubert/VariationalAnalysisWorkshop/index.html

10.1.3. Journal

10.1.3.1. Member of the Editorial Boards

- S. Gaubert is member of the editorial committee of the collection Mathématiques et Applications, SMAI and Springer.
- S. Gaubert is associate editor of Linear and Multilinear Algebra.
- S. Gaubert is associate editor of RAIRO Operations research.

10.1.4. Invited Talks

- M. Akian
  - Majorization inequalities for valuations of eigenvalues, Mittag-Leffler institute, February 2018.

- X. Allamigeon
  - Log-barrier interior point methods are not strongly polynomial, Workshop on Tropical Algebra and Applications, Mittag-Leffler Institute, Stockholm, Sweden, January 2018
  - Log-barrier interior point methods are not strongly polynomial, Workshop on Algebraic and geometric aspects of semidefinite programming at ISMP, Bordeaux, France, June 2018
  - Performance evaluation of an emergency call center: tropical polynomial systems applied to timed Petri nets, Workshop on Tropical Mathematics & Optimisation for Railways, Birmingham University, UK, June 2018
  - First steps in the formalization of convex polyhedra in Coq, International Conference on Mathematical Software, Notre-Dame University, USA, July 2018
- S. Gaubert
  - Nonarchimedean convex programming and its relation to mean payoff games, invited plenary talk at Highlights of Logic, Automata, and Games, Berlin, September 2018.
  - Condition numbers in nonarchimedean semidefinite programming and their relation with stochastic mean payoff games, Mittag-Leffler institute, February 2018.
  - Inequalities for the spectral radii and spectral norms of nonnegative tensors, ISMP, Bordeaux, Invited session, July 2018.
  - Nonarchimedean Degenerations of Convex Programming – In Which Tropical Geometry Shows That Log-Barrier Interior Point Methods Are Not Strongly Polynomial, Computational and Applied Mathematics Colloquium, The University of Chicago, May 2018.

10.1.5. Leadership within the Scientific Community

- See Section 10.1.1.1 (coordination of PGMO).

10.1.6. Research Administration

- M. Akian :
  - Member of the "comité de liaison SMAI-MODE" since June 2015.
  - Member of the laboratory council of CMAP.
- S. Gaubert :
  - Chairman of the Gaspard Monge Program for Optimization, Operations Research and their interactions with data sciences (PGMO), see 10.1.1.1 for details.
  - Member of the scientific council of CMAP.
- X. Allamigeon:
  - Member of the scientific committee of Inria Saclay – Ile-de-France.
  - Member of the Applied Mathematics Department committee at Ecole Polytechnique.

10.2. Teaching - Supervision - Juries

10.2.1. Teaching

- M. Akian
  - Course "Markov decision processes: dynamic programming and applications" joint between (3rd year of) ENSTA and M2 "Mathématiques et Applications", U. Paris Saclay, "Optimization", and shared with Jean-Philippe Chancelier (ENPC), 15 hours each.
- X. Allamigeon
  - Petites classes et encadrement d'enseignements d'approfondissement de Recherche Opérationnelle en troisième année à l'École Polytechnique (programme d'approfondissement de Mathématiques Appliquées) (niveau M1).

  - Cours du M2 “Optimisation” de l’Université Paris Saclay, cours partagé avec Manuel Ruiz (RTE) et Dominique Quadri (LRI, Université Paris Sud).
  - Co-responsabilité du programme d’approfondissement en mathématiques appliquées (troisième année) à l’École Polytechnique.
- J.B. Eytard
  - Cours de niveau L1-L2 à l’IUT d’Informatique d’Orsay (Univ. Paris-Sud), dans le cadre d’un monitorat (64h) (théorie des graphes, recherche opérationnelle, modélisation mathématique).
- S. Gaubert
  - Course “Systèmes à Événements Discrets”, option MAREVA, ENSMP.
  - Course “Algèbre tropicale pour le contrôle optimal et les jeux” of “Contrôle, Optimisation et Calcul des Variations” (COCV) of M2 “Mathématiques et Applications” of Paris 6 University and École Polytechnique.
  - Lecture of Operations Research, third year of École Polytechnique. The lectures notes were published as a book [66].
- M. Skomra
  - TD de mathématiques à l’UPMC.

10.2.2. Supervision

- PhD : Eric Fodjo, registered at École Polytechnique, since October 2013, thesis supervisor: Marianne Akian, defended on Jul 13 2017.
- PhD : Mateusz Skomra, registered at Univ. Paris Saclay since October 2015, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon, defended on Dec. 5, 2018.
- PhD : Jean-Bernard Eytard, registered at Univ. Paris Saclay since October 2015, thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian, Mustapha Bouhtou, defended on Nov. 12, 2018.
- PhD in progress: Paulin Jacquot, registered at Univ. Paris Saclay since November 2016, thesis supervisor: Stéphane Gaubert, cosupervision: Nadia Oujdane, Olivier Beaude (EDF).
- PhD in progress: Benoît Tran, registered at Univ Paris-Est Marne La Vallée, since September 2017, thesis supervisor: Jean-Philippe Chancelier (ENPC), cosupervision: Marianne Akian.
- PhD in progress: Maxime Grangereau, registered at Univ. Paris Saclay since Jan 2018, thesis supervisor: Emanuel Gobet, cosupervision: Stéphane Gaubert.
- PhD in progress: Omar Saadi, registered at Univ. Paris Saclay since October 2018, thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian.
- PhD in progress: Marin Boyet, registered at Univ. Paris Saclay since October 2018, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.

10.2.3. Juries

- M. Akian
  - Jury of the 2018 competition for CR2 positions of Inria Saclay–Île-de-France.
  - Jury of the PhD thesis of E. Fodjo (Ecole Polytechnique, thesis supervisor, July 2018).
  - Jury of the PhD thesis of H. Gerard (ENPC, reviewer, Oct. 2018).
  - Jury of the PhD thesis of J.B. Eytard (Ecole Polytechnique, examiner, Nov. 2018).
- X. Allamigeon
  - Comité de suivi doctoral de D. Rouhling (Inria Sophia, Mai 2018)
  - Jury of the PhD thesis of M. Skomra (Ecole Polytechnique, examiner, Dec. 2018).
- S. Gaubert

  - Jury of the HdR of Welington de Oliveira (Sorbonne Universités –Paris I– examiner, Déc. 2018)
  - Jury of the PhD thesis of J.B. Eytard (Ecole Polytechnique, examiner, Nov. 2018).
  - Jury of the PhD thesis of T. Roget (Ecole Polytechnique, examiner, Nov. 2018).
  - Jury of the PhD thesis of M. Skomra (Ecole Polytechnique, examiner, Dec. 2018).

10.3. Conferences, Seminars

- M. Akian
  - The operator approach to entropy games, International workshop on game theory, IHP, June 2018, Paris.
  - Majorization inequalities for valuations of eigenvalues, ANR/DFG PRCI Project Kick-off Meeting, July 2018, Bonn.
  - Tropical geometry, Optimal Control and Mean-payoff Games, Dagstuhl Seminar "Shape Analysis: Euclidean, Discrete and Algebraic Geometric Methods", Oct. 2018, Dagstuhl.
- X. Allamigeon
  - Tropical Linear Optimization, Polytopes à Paris, Paris, France, January 2018.
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