

RESEARCH CENTRE
Saclay - Île-de-France

2021
ACTIVITY REPORT

Project-Team
TROPICAL

**Tropical methods: structures, algorithms
and interactions**

IN COLLABORATION WITH: Centre de Mathématiques Appliquées
(CMAP)

DOMAIN

**Applied Mathematics, Computation and
Simulation**

THEME

**Optimization and control of dynamic
systems**

Contents

Project-Team TROPICAL	1
1 Team members, visitors, external collaborators	2
2 Overall objectives	3
2.1 Scientific context	3
3 Research program	4
3.1 Optimal control and zero-sum games	4
3.2 Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry	4
3.3 Tropical algebra and convex geometry	5
3.4 Tropical methods applied to optimization, perturbation theory and matrix analysis	5
4 Application domains	5
4.1 Discrete event systems (manufacturing systems, networks, emergency call centers)	5
4.2 Optimal control and games	6
4.3 Operations Research	6
4.4 Computing program and dynamical systems invariants	6
5 Social and environmental responsibility	6
5.1 Impact of research results	6
6 Highlights of the year	6
6.1 Awards	6
7 New software and platforms	7
7.1 New software	7
7.1.1 Coq-Polyhedra	7
7.1.2 EmergencyEval	7
8 New results	8
8.1 Optimal control and zero-sum games	8
8.1.1 Multiply Accelerated Value Iteration Algorithms For Classes of Markov Decision Processes	8
8.1.2 Polyhedral representation of multi-stage stochastic linear problems	8
8.1.3 Highway hierarchies for Hamilton-Jacobi-Bellman (HJB) PDEs	9
8.2 Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry	9
8.2.1 Volume in Hilbert and Funk geometries	9
8.2.2 Intrinsic characterization of Hilbert geometries	10
8.2.3 Reflexion and refraction in Finsler geometry	10
8.2.4 Barycentres in Funk/Hilbert geometries	10
8.2.5 Funk and Hilbert geometries of tropical convex sets	10
8.2.6 Firm non-expansive mappings	11
8.2.7 Metric functionals and invariant spaces	11
8.3 Tropical algebra and convex geometry	11
8.3.1 Formalizing convex polyhedra in Coq	11
8.3.2 Linear algebra over systems	12
8.3.3 Ambitropical convexity and Shapley retracts	12
8.3.4 Tropical linear regression and applications	12
8.3.5 Eigenvalues of Tropical Symmetric Matrices	12
8.3.6 Tropical Systems of Polynomial Equations	13
8.4 Tropical methods applied to optimization, perturbation theory and matrix analysis	13
8.4.1 Tropicalization of interior point methods and application to complexity	13
8.4.2 Tropical Nash equilibria and complementarity problems	13
8.5 Algebraic aspects of tensors and neural networks	14

8.5.1	Topology of tropical ranks	14
8.5.2	Approximation theory of neural networks	14
8.6	Applications	14
8.6.1	Performance evaluation of emergency call centers	14
8.6.2	Optimal control of energy flexibilities in a stochastic environment	15
8.6.3	Bilevel programming applied to optimal pricing of energy contracts	15
8.6.4	POMDP and adaptive management	15
9	Bilateral contracts and grants with industry	16
9.1	Bilateral contracts with industry	16
10	Partnerships and cooperations	16
10.1	International initiatives	16
10.1.1	STIC/MATH/CLIMAT AmSud project	16
10.1.2	Bilateral project FACCTS with the University of Chicago	16
10.2	International research visitors	17
10.2.1	Visits to international teams	17
10.3	National initiatives	17
10.3.1	ANR	17
10.3.2	Programme Gaspard Monge pour l'optimisation, la recherche opérationnelle et leurs interactions avec les sciences des données	17
10.3.3	Centre des Hautes Études du Ministère de l'Intérieur	17
11	Dissemination	17
11.1	Promoting scientific activities	17
11.1.1	Scientific events: organisation	17
11.1.2	Scientific events: selection	18
11.1.3	Journal	18
11.1.4	Invited talks	18
11.1.5	Leadership within the scientific community	19
11.1.6	Scientific expertise	19
11.1.7	Research administration	19
11.2	Teaching - Supervision - Juries	19
11.2.1	Teaching	19
11.2.2	Supervision	20
11.2.3	Juries	21
11.3	Conferences, Seminars	21
12	Scientific production	22
12.1	Major publications	22
12.2	Publications of the year	23
12.3	Cited publications	25

Project-Team TROPICAL

Creation of the Project-Team: 2018 July 01

Keywords

Computer sciences and digital sciences

- A1.2.4. – QoS, performance evaluation
- A2.3.3. – Real-time systems
- A2.4. – Formal method for verification, reliability, certification
- A6.2.5. – Numerical Linear Algebra
- A6.2.6. – Optimization
- A6.4.2. – Stochastic control
- A6.4.6. – Optimal control
- A7.2.4. – Mechanized Formalization of Mathematics
- A8.1. – Discrete mathematics, combinatorics
- A8.3. – Geometry, Topology
- A8.9. – Performance evaluation
- A8.11. – Game Theory
- A9.6. – Decision support

Other research topics and application domains

- B4.3. – Renewable energy production
- B4.4. – Energy delivery
 - B4.4.1. – Smart grids
- B6.6. – Embedded systems
- B8.4. – Security and personal assistance
 - B8.4.1. – Crisis management

1 Team members, visitors, external collaborators

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2 Overall objectives

The project develops tropical methods motivated by applications arising in decision theory (deterministic and stochastic optimal control, game theory, optimization and operations research), in the analysis or control of classes of dynamical systems (including timed discrete event systems and positive systems), in the verification of programs and systems, and in the development of numerical algorithms. Tropical algebra tools are used in interaction with various methods, coming from convex analysis, Hamilton–Jacobi partial differential equations, metric geometry, Perron-Frobenius and nonlinear fixed-point theories, combinatorics or algorithmic complexity. The emphasis of the project is on mathematical modelling and computational aspects.

The subtitle of the Tropical project, namely, “structures, algorithms, and interactions”, refers to the spirit of our research, including a methodological component, computational aspects, and finally interactions with other scientific fields or real world applications, in particular through mathematical modelling.

2.1 Scientific context

Tropical algebra, geometry, and analysis have enjoyed spectacular development in recent years. Tropical structures initially arose to solve problems in performance evaluation of discrete event systems [49], combinatorial optimization [54], or automata theory [85]. They also arose in mathematical physics and asymptotic analysis [78, 75]. More recently, these structures have appeared in several areas of pure mathematics, in particular in the study of combinatorial aspects of algebraic geometry [69, 96, 88, 74], in algebraic combinatorics [64], and in arithmetics [58]. Also, further applications of tropical methods have appeared, including optimal control [79], program invariant computation [43] and timed systems verification [77], and zero-sum games [1].

The term ‘tropical’ generally refers to algebraic structures in which the laws originate from optimization processes. The prototypical tropical structure is the max-plus semifield, consisting of the real numbers, equipped with the maximum, thought of as an additive law, and the addition, thought of as a multiplicative law. Tropical objects appear as limits of classical objects along certain deformations (“log-limits sets” of Bergman, “Maslov dequantization”, or “Viro deformation”). For this reason, the introduction of tropical tools often yields new insights into old familiar problems, leading either to counterexamples or to new methods and results; see for instance [96, 82]. In some applications, like optimal control, discrete event systems, or static analysis of programs, tropical objects do not appear through a limit procedure, but more directly as a modelling or computation/analysis tool; see for instance [93, 49, 72, 55].

Tropical methods are linked to the fields of positive systems and of metric geometry [84], [12]. Indeed, tropically linear maps are monotone (a.k.a. order-preserving). They are also nonexpansive in certain natural metrics (sup-norm, Hopf oscillation, Hilbert’s projective metric, ...). In this way, tropical dynamical systems appear to be special cases of nonexpansive, positive, or monotone dynamical systems, which are studied as part of linear and non-linear Perron-Frobenius theory [76], [2]. Such dynamical systems are of fundamental importance in the study of repeated games [81]. Monotonicity properties are also essential in the understanding of the fixed points problems which determine program invariants by abstract interpretation [59]. The latter problems are actually somehow similar to the ones arising in the study of zero-sum games; see [6]. Moreover, positivity or monotonicity methods are useful in population dynamics, either in a discrete space setting [94] or in a PDE setting [51]. In such cases, solving tropical problems often leads to solutions or combinatorial insights on classical problems involving positivity conditions (e.g., finding equilibria of dynamical systems with nonnegative coordinates, understanding the qualitative and quantitative behavior of growth rates / Floquet eigenvalues [10], etc). Other applications of Perron-Frobenius theory originate from quantum information and control [87, 92].

3 Research program

3.1 Optimal control and zero-sum games

The dynamic programming approach allows one to analyze one or two-player dynamic decision problems by means of operators, or partial differential equations (Hamilton–Jacobi or Isaacs PDEs), describing the time evolution of the value function, i.e., of the optimal reward of one player, thought of as a function of the initial state and of the horizon. We work especially with problems having long or infinite horizon, modelled by stopping problems, or ergodic problems in which one optimizes a mean payoff per time unit. The determination of optimal strategies reduces to solving nonlinear fixed point equations, which are obtained either directly from discrete models, or after a discretization of a PDE.

The geometry of solutions of optimal control and game problems Basic questions include, especially for stationary or ergodic problems, the understanding of existence and uniqueness conditions for the solutions of dynamic programming equations, for instance in terms of controllability or ergodicity properties, and more generally the understanding of the structure of the full set of solutions of stationary Hamilton–Jacobi PDEs and of the set of optimal strategies. These issues are already challenging in the one-player deterministic case, which is an application of choice of tropical methods, since the Lax–Oleinik semigroup, i.e., the evolution semigroup of the Hamilton–Jacobi PDE, is a linear operator in the tropical sense. Recent progress in the deterministic case has been made by combining dynamical systems and PDE techniques (weak KAM theory [61]), and also using metric geometry ideas (abstract boundaries can be used to represent the sets of solutions [73], [4]). The two player case is challenging, owing to the lack of compactness of the analogue of the Lax–Oleinik semigroup and to a richer geometry. The conditions of solvability of ergodic problems for games (for instance, solvability of ergodic Isaacs PDEs), and the representation of solutions are only understood in special cases, for instance in the finite state space case, through tropical geometry and non-linear Perron–Frobenius methods [40], [41], [2].

Algorithmic aspects: from combinatorial algorithms to the attenuation of the curse of dimensionality Our general goal is to push the limits of solvable models by means of fast algorithms adapted to large scale instances. Such instances arise from discrete problems, in which the state space may so large that it is only accessible through local oracles (for instance, in some web ranking applications, the number of states may be the number of web pages) [62]. They also arise from the discretization of PDEs, in which the number of states grows exponentially with the number of degrees of freedom, according to the “curse of dimensionality”. A first line of research is the development of *new approximation methods for the value function*. So far, classical approximations by linear combinations have been used, as well as approximation by suprema of linear or quadratic forms, which have been introduced in the setting of dual dynamic programming and of the so called “max-plus basis methods” [63]. We believe that more concise or more accurate approximations may be obtained by unifying these methods. Also, some max-plus basis methods have been shown to *attenuate the curse of dimensionality* for very special problems (for instance involving switching) [80, 66]. This suggests that the complexity of control or games problems may be measured by more subtle quantities than the mere number of states, for instance, by some forms of metric entropy (for example, certain large scale problems have a low complexity owing to the presence of decomposition properties, “highway hierarchies”, etc.). A second line of our research is the development of *combinatorial algorithms*, to solve large scale zero-sum two-player problems with discrete state space. This is related to current open problems in algorithmic game theory. In particular, the existence of polynomial-time algorithms for games with ergodic payment is an open question. See e.g. [44] for a polynomial time average complexity result derived by tropical methods. The two lines of research are related, as the understanding of the geometry of solutions allows to develop better approximation or combinatorial algorithms.

3.2 Non-linear Perron–Frobenius theory, nonexpansive mappings and metric geometry

Several applications (including population dynamics [10] and discrete event systems [49, 57, 45]) lead to studying classes of dynamical systems with remarkable properties: preserving a cone, preserving an order, or being nonexpansive in a metric. These can be studied by techniques of non-linear Perron–Frobenius

theory [2] or metric geometry [11]. Basic issues concern the existence and computation of the “escape rate” (which determines the throughput, the growth rate of the population), the characterizations of stationary regimes (non-linear fixed points), or the study of the dynamical properties (convergence to periodic orbits). Nonexpansive mappings also play a key role in the “operator approach” to zero-sum games, since the one-day operators of games are nonexpansive in several metrics, see [8].

3.3 Tropical algebra and convex geometry

The different applications mentioned in the other sections lead us to develop some basic research on tropical algebraic structures and in convex and discrete geometry, looking at objects or problems with a “piecewise-linear” structure. These include the geometry and algorithmics of tropical convex sets [48, 1], tropical semialgebraic sets [32], the study of semi-modules (analogues of vector spaces when the base field is replaced by a semi-field), the study of systems of equations linear in the tropical sense, investigating for instance the analogues of the notions of rank, the analogue of the eigenproblems [42], and more generally of systems of tropical polynomial equations. Our research also builds on, and concerns, classical convex and discrete geometry methods.

3.4 Tropical methods applied to optimization, perturbation theory and matrix analysis

Tropical algebraic objects appear as a deformation of classical objects through various asymptotic procedures. A familiar example is the rule of asymptotic calculus,

$$e^{-a/\epsilon} + e^{-b/\epsilon} \asymp e^{-\min(a,b)/\epsilon}, \quad e^{-a/\epsilon} \times e^{-b/\epsilon} = e^{-(a+b)/\epsilon}, \quad (1)$$

when $\epsilon \rightarrow 0^+$. Deformations of this kind have been studied in different contexts: large deviations, zero-temperature limits, Maslov’s “dequantization method” [78], non-archimedean valuations, log-limit sets and Viro’s patchworking method [96], etc.

This entails a relation between classical algorithmic problems and tropical algorithmic problems, one may first solve the $\epsilon = 0$ case (non-archimedean problem), which is sometimes easier, and then use the information gotten in this way to solve the $\epsilon = 1$ (archimedean) case.

In particular, tropicalization establishes a connection between polynomial systems and piecewise affine systems that are somehow similar to the ones arising in game problems. It allows one to transfer results from the world of combinatorics to “classical” equations solving. We investigate the consequences of this correspondence on complexity and numerical issues. For instance, combinatorial problems can be solved in a robust way. Hence, situations in which the tropicalization is faithful lead to improved algorithms for classical problems. In particular, scalings for the polynomial eigenproblems based on tropical preprocessings have started to be used in matrix analysis [67, 71].

Moreover, the tropical approach has been recently applied to construct examples of linear programs in which the central path has an unexpectedly high total curvature [7], and it has also led to positive polynomial-time average case results concerning the complexity of mean payoff games. Similarly, we are studying semidefinite programming over non-archimedean fields [32], [47], with the goal to better understand complexity issues in classical semidefinite and semi-algebraic programming.

4 Application domains

4.1 Discrete event systems (manufacturing systems, networks, emergency call centers)

One important class of applications of max-plus algebra comes from discrete event dynamical systems [49]. In particular, modelling timed systems subject to synchronization and concurrency phenomena leads to studying dynamical systems that are non-smooth, but which have remarkable structural properties (nonexpansiveness in certain metrics, monotonicity) or combinatorial properties. Algebraic methods allow one to obtain analytical expressions for performance measures (throughput, waiting time, etc). A recent application, to emergency call centers, can be found in [45].

4.2 Optimal control and games

Optimal control and game theory have numerous well established applications fields: mathematical economy and finance, stock optimization, optimization of networks, decision making, etc. In most of these applications, one needs either to derive analytical or qualitative properties of solutions, or design exact or approximation algorithms adapted to large scale problems.

4.3 Operations Research

We develop, or have developed, several aspects of operations research, including the application of stochastic control to optimal pricing, optimal measurement in networks [89]. Applications of tropical methods arise in particular from discrete optimization [55], [56], scheduling problems with and-or constraints [83], or product mix auctions [95].

4.4 Computing program and dynamical systems invariants

A number of programs and systems verification questions, in which safety considerations are involved, reduce to computing invariant subsets of dynamical systems. This approach appears in various guises in computer science, for instance in static analysis of program by abstract interpretation, along the lines of P. and R. Cousot [59], but also in control (eg, computing safety regions by solving Isaacs PDEs). These invariant sets are often sought in some tractable effective class: ellipsoids, polyhedra, parametric classes of polyhedra with a controlled complexity (the so called “templates” introduced by Sankaranarayanan, Sipma and Manna [91]), shadows of sets represented by linear matrix inequalities, disjunctive constraints represented by tropical polyhedra [43], etc. The computation of invariants boils down to solving large scale fixed point problems. The latter are of the same nature as the ones encountered in the theory of zero-sum games, and so, the techniques developed in the previous research directions (especially methods of monotonicity, nonexpansiveness, discretization of PDEs, etc) apply to the present setting, see e.g. [65, 68] for the application of policy iteration type algorithms, or for the application for fixed point problems over the space of quadratic forms [6]. The problem of computation of invariants is indeed a key issue needing the methods of several fields: convex and nonconvex programming, semidefinite programming and symbolic computation (to handle semialgebraic invariants), nonlinear fixed point theory, approximation theory, tropical methods (to handle disjunctions), and formal proof (to certify numerical invariants or inequalities).

5 Social and environmental responsibility

5.1 Impact of research results

The team has developed collaborations on the dimensioning of emergency call centers, with Préfecture de Police (Plate Forme d’Appels d’Urgence - PFAU - 17-18-112, operated jointly by Brigade de sapeurs pompiers de Paris and by Direction de la sécurité de proximité de l’agglomération parisienne) and also with the Emergency medical services of Assistance Publique – Hôpitaux de Paris (Centre 15 of SAMU75, 92, 93 and 94). This work is described further in Section 8.6.1.

6 Highlights of the year

6.1 Awards

X. Allamigeon, P. Benchimol and S. Gaubert (from the Tropical team) and M. Joswig (from TU Berlin) obtained the SIAM Sigest Award in 2021, for the work “Log-barrier interior point methods are not strongly polynomial”, SIAM Journal on Applied Algebra and Geometry 2, 1, 2018, p. 140–178, reprinted in extended form in the SIGEST section of SIAM Review, see: What Tropical Geometry Tells Us about the Complexity of Linear Programming, SIAM Review 63, 1, February 2021, p. 123–164, [14]. According to SIAM, “The purpose of SIGEST is to make the 10,000+ readers of SIAM Review aware of exceptional papers published in SIAM’s specialized journals.”

7 New software and platforms

7.1 New software

7.1.1 Coq-Polyhedra

Name: Coq-Polyhedra

Keywords: Coq, Polyhedra, Automated theorem proving, Linear optimization

Scientific Description: Coq-Polyhedra is a library providing a formalization of convex polyhedra in the Coq proof assistant. While still in active development, it provides an implementation of the simplex method, and already handles the basic properties of polyhedra such as emptiness, boundedness, membership. Several fundamental results in the theory of convex polyhedra, such as Farkas Lemma, duality theorem of linear programming, and Minkowski Theorem, are also formally proved.

The formalization is based on the Mathematical Components library, and makes an extensive use of the boolean reflection methodology.

Functional Description: Coq-Polyhedra is a library which aims at formalizing convex polyhedra in Coq

News of the Year: Coq-Polyhedra now provides most of the basic operations on polyhedra. They are expressed on a quotient type that avoids reasoning with particular inequality representations. They include : * the construction of elementary polyhedra (half-spaces, hyperplanes, affine spaces, orthants, simplices, etc) * basic operations such as intersection, projection (thanks to the formalization of the Fourier-Motzkin algorithm), image under linear functions, computations of convex hulls, finitely generated cones, etc. * computation of affine hulls of polyhedra, as well as their dimension

Thanks to this, we have made huge progress on the formalization of the combinatorics of polyhedra. The poset of faces, as well as its fundamental properties (lattice, gradedness, atomicity and co-atomicity, etc) are now formalized. The manipulation of the faces is based on an extensive use of canonical structures, that allows to get the most appropriate inequality representations for reasoning. In this way, we arrive at very concise and elegant proofs, closer to the pen-and-paper ones.

URL: <https://github.com/nhojem/Coq-Polyhedra>

Publications: hal-01673390, hal-03151656, hal-03505724, hal-01967575, hal-01967576

Contact: Xavier Allamigeon

Participants: Xavier Allamigeon, Vasileios Charisopoulos, Quentin Canu, Ricardo Katz, Pierre-Yves Strub

Partners: CIFASIS, Ecole Polytechnique

7.1.2 EmergencyEval

Keywords: Dynamic Analysis, Simulation, Ocaml, Emergency, Firefighters, Police

Scientific Description: This software aims at enabling the definition of a Petri network execution semantic, as well as the instantiation and execution of said network using the aforedefined semantic.

The heart of the project dwells in its kernel which operates the step-by-step execution of the network, obeying rules provided by an oracle. This user-defined and separated oracle computes the information necessary to the kernel for building the next state using the current state. The base of our software is the framework for the instantiation and execution of Petri nets, without making assumptions regarding the semantic.

In the context of the study of the dynamics of emergency call centers, a second part of this software is the definition and implementation of the semantic of call centers modeled as Petri nets, and more specifically timed prioritized Petri nets. A module interoperating with the kernel enables to include all the operational specificities of call centers (urgency level, discriminating between

operators and callers ...) while guaranteeing the genericity of the kernel which embeds the Petri net formalism as such.

Functional Description: In order to enable the quantitative study of the throughput of calls managed by emergency center calls and the assessment of various organisational configurations considered by the stakeholders (firefighters, police, medical emergency service of the 75, 92, 93 and 94 French departments), this software modelizes their behaviours by resorting to extensions of the Petri net formalism. Given a call transfer protocol in a call center, which corresponds to a topology and an execution semantic of a Petri net, the software generates a set of entering calls in accord with the empirically observed statistic distributions (share of very urgent calls, conversation length), then simulates its management by the operators with respect to the defined protocol. Transitional regimes phenomena (peak load, support) which are not yet handled by mathematical analysis could therefore be studied. The output of the software is a log file which is an execution trace of the simulation featuring extensive information in order to enable the analysis of the data for providing simulation-based insights for decision makers.

The software relies on a Petri net simulation kernel designed to be as modular and adaptable as possible, fit for simulating other Petri-net related phenomena, even if their semantic differ greatly.

Contact: Xavier Allamigeon

Participants: Xavier Allamigeon, Benjamin Nguyen-Van-Yen, Baptiste Colin

8 New results

8.1 Optimal control and zero-sum games

8.1.1 Multiply Accelerated Value Iteration Algorithms For Classes of Markov Decision Processes

Participants: Marianne Akian, Stéphane Gaubert, Omar Saadi.

Accelerated gradient algorithms in convex optimization were introduced by Nesterov. A fundamental question is whether similar acceleration schemes work for the iteration of nonexpansive mappings. In a joint work with Zheng Qu (Hong Kong University) [13], motivated by the analysis of Markov decision processes and zero-sum repeated games, we study fixed point problems for Shapley operators, i.e., for sup-norm nonexpansive and order preserving mapping. We deal more especially with affine operators, corresponding to zero-player problems – the latter can be used as a building block for one or two player problems, by means of policy iteration. For an affine operator, associated to a Markov chain, the acceleration property can be formalized as follows: one should replace an original scheme with a convergence rate $1 - \Theta(\epsilon)$ by a convergence rate $1 - \Theta(\epsilon^{1/2})$ where ϵ is the spectral gap of the Markov chain. We characterize the spectra of Markov chains for which this acceleration is possible. We also characterize the spectra for which a multiple acceleration is possible, leading to a rate of $1 - \Theta(\epsilon^{1/d})$ for $d > 2$.

8.1.2 Polyhedral representation of multi-stage stochastic linear problems

Participants: Maël Forcier, Stéphane Gaubert.

In [25] (joint work with Vincent Leclère, ENPC), we study multistage stochastic problems with a linear structure and general cost distribution. We obtain an exact quantization result, showing that a multistage problem with a continuous cost distribution is equivalent to a problem with a discrete distribution, constructed by exploiting results from polyhedral geometry (“nested” analogue of fiber polytopes). We deduce polynomial-time solvability results in fixed dimension, both for exact and approximated versions of the problem.

8.1.3 Highway hierarchies for Hamilton-Jacobi-Bellman (HJB) PDEs

Participants: Marianne Akian, Stéphane Gaubert, Shanqing Liu.

Hamilton-Jacobi-Bellman equations arise as the dynamic programming equations of deterministic or stochastic optimal control problems. They allow to obtain the global optimum of these problems and to synthesize an optimal feedback control, leading to a solution robust against system perturbations. Several methods have been proposed in the literature to bypass the obstruction of curse of dimensionality of such equations, assuming a certain structure of the problem, and/or using “unstructured discretizations”, that are not based on given grids. Among them, one may cite tropical numerical method, and probabilistic numerical method. On another direction, “highway hierarchies”, developed by Sanders, Schultes and coworkers [60, 90], initially for applications to on-board GPS systems, are a computational method that allows one to accelerate Dijkstra algorithm for discrete time and state shortest path problems.

The aim of the PhD thesis of Shanqing Liu is to develop new numerical methods to solve Hamilton-Jacobi-Bellman equations that are less sensitive to curse of dimensionality.

In a first work, we have developed a multilevel fast-marching method, extending to the PDE case the idea of “highway hierarchies”. Given the problem of finding an optimal trajectory between two given points, the method consists in refining the grid only in a neighborhood of the optimal trajectory, which is itself computed using an approximation of the value function on a coarse grid. A first account of this work has been presented to the PGMO days in December 2021.

8.2 Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

8.2.1 Volume in Hilbert and Funk geometries

Participants: Constantin Vernicos, Cormac Walsh.

In a recent paper [19], we investigated how the volume of a ball in a Hilbert geometry grows as its radius increases. In particular, we studied the volume entropy

$$\lim_{r \rightarrow \infty} \frac{\log \text{Vol} B(x, r)}{r},$$

where $B(x, r)$ is the metric ball with center x and radius r , and Vol denotes the Holmes–Thompson volume. Note that the volume entropy does not depend on the particular choice of x . We showed that the volume entropy is exactly twice the flag-approximability of the convex body. This is a new notion of approximability we introduced that measures the complexity of a polytope by counting its number of flags rather than its number of vertices. A corollary is that the Euclidean ball has the maximal volume entropy among Hilbert geometries of a given dimension, a fact that was recently proved by Tholozan by other means. We also showed that the rate of growth of the volume is minimised when the convex body is a simplex.

We are continuing this work by investigating the volume of balls of finite radius, rather than the asymptotics. We have found it convenient to turn our attention to a metric different from the Hilbert metric, but related to it. The Funk metric, as it is called, lacks the symmetry property usually assumed for metric spaces. However, it is somewhat simpler to work with when dealing with volumes, and it exhibits the same interesting behaviour. Given a convex body and a radius r , there is a unique point x such that the Funk ball of radius r centered at x minimises the volume over all Funk balls of the same radius. It is natural to conjecture that this minimum volume, which depends on the convex body, is maximised when the body is a Euclidean ball. If this is true, one could recover Blaschke–Santaló inequality by letting the radius tend to zero, and the centro-affine isoperimetric inequality by letting the radius tend to infinity. Similarly, consideration of the minimum provides an interpolation between the Mahler conjecture and Kalai’s flag conjecture.

We have been studying in more detail the volume of balls in the Funk geometry when the convex body is a polytope. Here, as in the case of the Hilbert metric, the volume grows polynomially with order equal to the dimension, and the constant on front of the highest order term depends only on the number of flags of the polytope. Thus, this term does not change when the polytope is perturbed in way that doesn't change the combinatorics. This motivates us to look at the second highest order term. We have developed a formula for this in terms of the position of the vertices of the polytope and the vertices of its dual. Thus we get a new centro-affine invariant for polytopes. We are in the process of studying this invariant, to see where it is maximised and minimised, etc.

8.2.2 Intrinsic characterization of Hilbert geometries

Participants: Constantin Vernicos.

This is a joint work with Antonin Guilloux (IMJ).

In a Hilbert geometry, if one knows the shape of a metric ball of radius R at some point, then one knows the geometry it lives in. However knowing the tangent unit ball at one point is not enough. In dimension one a simple computation shows that one needs to know the tangent ball at two points. Hence we conjecture that in dimension n one needs to know the shape of tangent ball at $n + 1$ points to be able to characterize the convex sets defining the geometry. We were able to prove the conjecture for polytopes in dimension 2, and we are currently looking at smooth convex sets and higher dimension.

8.2.3 Reflexion and refraction in Finsler geometry

Participants: Constantin Vernicos.

The law of reflection in a smooth strictly convex Finsler geometry was established by Gutkin and Tabachnikov who were studying billiards in that setting. Establishing it in the non-smooth not-strictly-convex case forced us to give an affine interpretation which also permits one to establish Snell-Descartes laws of refraction. We also get to give an affine interpretation of refraction with negative indices. We are investigating existence of periodic orbits in that setting.

8.2.4 Barycentres in Funk/Hilbert geometries

Participants: Constantin Vernicos.

The existence of certain type of Barycentre in a metric geometry is related to certain types of curvature. For instance in Hyperbolic geometry the fact that the square of the distance is strictly convex gives existence, and is the founding fact in the barycentric methods introduced by Besson-Courtois-Gallot, which allowed them to prove the minimal entropy theorem. We are investigating the existence of such barycentres in the setting of Hilbert geometries, which share some similarity with the hyperbolic geometries, but whose distance is never convex in the sense of Busemann, which is a weaker assumption than what happens in Hyperbolic geometry.

8.2.5 Funk and Hilbert geometries of tropical convex sets

Participants: Constantin Vernicos, Stéphane Gaubert.

We investigate an analogue of Funk and Hilbert geometries inside tropical convex sets. This is motivated by the construction of barriers adapted to tropical polyhedra, and by nonarchimedean convexity.

8.2.6 Firm non-expansive mappings

Participants: Armando Gutiérrez, Cormac Walsh.

A fundamental question in the theory of metric spaces is the long term behaviour of iterates of non-expansive mappings. Particularly interesting is the case where the mapping has no fixed point, because here the iterates have the possibility of escaping to infinity.

In [30], we introduce the notion of *firm non-expansive mapping* in an arbitrary (weak) metric space. We show that that, for these mappings, the minimal displacement, the linear rate of escape, and the asymptotic step size are all equal.

This generalises a previous result of Reich and Shafrir in the setting of Banach spaces, and also one by Ariza-Ruiz et al. in the setting of “W-hyperbolic spaces”, which are geodesic metric spaces with a certain “negative curvature”-type condition.

The advantage of our definition is that we do not need to assume the existence of geodesics. This is significant since in modern optimisation applications one often deals with discrete spaces or has access to a collection of points of a space whose geometric structure is unknown. Our class of firm non-expansive mappings includes the mappings considered in the setting of Banach spaces or W-hyperbolic spaces.

Our definition of firm non-expansive was inspired by Ćirić’s work on generalisations of Banach’s Contraction Theorem, for which he introduced the notion of *generalised contraction*. The relation between this concept and our firm non-expansive mappings is similar to the relation between strict contractions and non-expansive mappings.

8.2.7 Metric functionals and invariant spaces

Participants: Armando Gutiérrez.

In a joint work with Anders Karlsson (University of Geneva and Uppsala University), we use the explicit formulas for metric functionals on ℓ_1 and on Hilbert spaces to provide a new result for non-expansive mappings in ℓ_1 and to study the well-known invariant subspace problem, respectively.

More precisely, the first result is that for every non-expansive mapping T in ℓ_1 , there exists a non-trivial continuous linear functional f such that $f(T^n 0) \geq 0$ for all $n \geq 0$. The second result is that for every affine non-expansive mapping $T = U + T0$ in a Hilbert space, if 0 is not an element of the norm closure of the set $\text{Ran}(I-T)$, then there exists a co-dimension one closed invariant subspace for the linear operator U .

8.3 Tropical algebra and convex geometry

8.3.1 Formalizing convex polyhedra in Coq

Participants: Xavier Allamigeon, Quentin Canu.

In a joint work with Ricardo Katz (Conicet, Argentina) and Pierre-Yves Strub (LIX, Ecole Polytechnique), we present the first formalization of faces of polyhedra in the proof assistant Coq. This builds on the formalization of a library providing the basic constructions and operations over polyhedra, including projections, convex hulls and images under linear maps. Moreover, we design a special mechanism which automatically introduces an appropriate representation of a polyhedron or a face, depending on the context of the proof. We demonstrate the usability of this approach by establishing some of the most important combinatorial properties of faces, namely that they constitute a family of graded atomistic and coatomistic lattices closed under sublattices. We also prove a theorem due to Balinski on the d -connectedness of the adjacency graph of polytopes of dimension d . This is implemented in the

CoqPolyhedra library (we refer to the software session for more details). This work has been published in 10th International Joint Conference on Automated Reasoning [35], and is currently accepted (under minor revisions) in a special issue of Logical Methods in Computer Science.

In collaboration with P. Y. Strub, X. Allamigeon and Q. Canu are currently working on the enumeration of vertices and the computation of graphs of polytopes within the proof assistant Coq. They have also contributed to the improvement of the implementation of hierarchies of mathematical objects in the MathComp library [21]. Another work on the formalization of lattices and ordered structures, with a special emphasis on finite (sub)lattices, is also currently ongoing.

8.3.2 Linear algebra over systems

Participants: Marianne Akian, Stéphane Gaubert.

In a joint work with Louis Rowen (Univ. Bar Ilan), we study linear algebra and convexity properties over “systems”. The latter provide a general setting encompassing extensions of the tropical semifields and hyperfields. Moreover, they have the advantage to be well adapted to the study of linear or polynomial equations.

8.3.3 Ambitropical convexity and Shapley retracts

Participants: Marianne Akian, Stéphane Gaubert.

Closed tropical convex cones are the most basic examples of modules over the tropical semifield. They coincide with sub-fixed-point sets of Shapley operators – dynamic programming operators of zero-sum games. We study a larger class of cones, which we call “ambitropical” as it includes both tropical cones and their duals. Ambitropical cones can be defined as lattices in the order induced by \mathbb{R}^n . Closed ambitropical cones are precisely the fixedpoint sets of Shapley operators. They are characterized by a property of best co-approximation arising from the theory of nonexpansive retracts of normed spaces. Finitely generated ambitropical cones arise when considering Shapley operators of deterministic games with finite action spaces. They are also special cases of hyperconvex spaces. Finitely generated ambitropical cones are special polyhedral complexes whose cells are alcoved polyhedra, and locally, they are in bijection with order preserving retracts of the Boolean cube. This is a joint work with Sara Vannucci. See [22].

8.3.4 Tropical linear regression and applications

Participants: Marianne Akian, Stéphane Gaubert, Yang Qi, Omar Saadi.

In [23], we show that the problem consisting in computing a best approximation of a collection of points by a tropical hyperplane is equivalent to solving a mean payoff game, and also, to compute the maximal radius of an inscribed ball in a tropical polytope. We provide an application to a problem of auction theory – measuring the distance to equilibrium. We also study a dual problem — computing the minimal radius of a circumscribed ball to a tropical polytope – and apply it to the rank-one approximation of tropical matrices and tensors.

8.3.5 Eigenvalues of Tropical Symmetric Matrices

Participants: Marianne Akian, Stéphane Gaubert, Hanieh Tavakolipour.

The tropical semifield can be thought of as the image of a field with a non-archimedean valuation. It allows in this way to study the asymptotics of Puiseux series with complex coefficients. When dealing with Puiseux series with real coefficients and with its associated order, it is convenient to use the symmetrized tropical semiring introduced in [86] (see also [49]), and the signed valuation which associates to any series its valuation together with its sign.

We study with these tools the asymptotics of eigenvalues and eigenvectors of symmetric positive definite matrices over the field of Puiseux series. This raises the problem of defining the appropriate notions of positive definite matrices over the symmetrized tropical semiring, eigenvalues and eigenvectors of such matrices, thus roots of polynomials and their multiplicities. This builds on [32] and [50].

8.3.6 Tropical Systems of Polynomial Equations

Participants: Marianne Akian, Antoine Bereau, Stéphane Gaubert

The PhD thesis of Antoine Bereau, started in September 2021, deals with systems of polynomial equations over tropical semifields. In the case of linear equations (that is polynomials with degree 1) [1] and in some other particular cases [34], the solution can be characterized as the value of a zero-sum deterministic or stochastic game, allowing one to use the associated algorithms. The aim of the thesis is to apply zero-sum games, combinatorial and polyhedral techniques, in order to solve tropical polynomial systems. One shall also use the extended and symmetrized tropical semirings which were already useful in the study of linear systems [86, 39]. A first investigation concerns an optimal degree bound for the linearization method based on the tropical analogue of the Macaulay matrix.

8.4 Tropical methods applied to optimization, perturbation theory and matrix analysis

8.4.1 Tropicalization of interior point methods and application to complexity

Participants: Xavier Allamigeon, Stéphane Gaubert, Nicolas Vandame.

It is an open question to determine if the theory of self-concordant barriers can provide an interior point method with strongly polynomial complexity in linear programming. In the special case of the logarithmic barrier, it was shown in [7, 14] that the answer is negative.

In a joint work [37] with Abdellah Aznag (Columbia University) and Yassine Hamdi (Ecole Polytechnique), we have studied the tropicalization of the central path associated with the entropic barrier studied by Bubeck and Eldan (Proc. Mach. Learn. Research, 2015), i.e., the logarithmic limit of this central path for a parametric family of linear programs defined over the field of Puiseux series. Our main result is that the tropicalization of the entropic central path is a piecewise linear curve which coincides with the tropicalization of the logarithmic central path studied by Allamigeon et al. in [7].

In a subsequent work, we have shown that *none* of the self-concordant barrier interior point methods is strongly polynomial. This result is obtained by establishing that, on parametric families of convex optimization problems, the log-limit of the central path degenerates to the same piecewise linear curve, independently of the choice of the barrier function. We provide an explicit linear program that falls in the same class as the Klee–Minty counterexample, i.e., a n -dimensional combinatorial cube, in which the number of iterations is 2^n .

8.4.2 Tropical Nash equilibria and complementarity problems

Participants: Xavier Allamigeon, Stéphane Gaubert.

Linear complementarity programming is a generalization of linear programming which encompasses the computation of Nash equilibria for bimatrix games. While the latter problem is PPAD-complete, we show in [38] that the analogue of this problem in tropical algebra can be solved in polynomial time. Moreover, we prove that the Lemke–Howson algorithm carries over the tropical setting and performs a linear number of pivots in the worst case. A consequence of this result is a new class of (classical) bimatrix games for which Nash equilibria computation can be done in polynomial time. This is joint work with Frédéric Meunier (Cermics, ENPC).

8.5 Algebraic aspects of tensors and neural networks

8.5.1 Topology of tropical ranks

Participants: Yang Qi.

The primary goal of [33] is to better understand the topological properties of various tensor ranks, which have useful practical implications.

In this ongoing project, we would like to study the corresponding problems for the space of tensors of a fixed Barvinok rank.

8.5.2 Approximation theory of neural networks

Participants: Yang Qi.

In [18] (joint work with Lek-Heng Lim and Mateusz Michałek) we show that the empirical risk minimization (ERM) problem for neural networks has no solution in general. More precisely, given a training set $s_1, \dots, s_n \in \mathbb{R}^p$ with corresponding responses $t_1, \dots, t_n \in \mathbb{R}^q$, fitting a k -layer neural network $v_\theta : \mathbb{R}^p \rightarrow \mathbb{R}^q$ involves estimation of the weights $\theta \in \mathbb{R}^m$ via an ERM:

$$\inf_{\theta \in \mathbb{R}^m} \sum_{i=1}^n \|t_i - v_\theta(s_i)\|_2^2.$$

We show that even for $k = 2$, this infimum is not attainable in general for common activations like ReLU, hyperbolic tangent, and sigmoid functions. In addition, we show that for smooth activations $\sigma(x) = 1/(1 + \exp(-x))$ and $\sigma(x) = \tanh(x)$, such failure to attain an infimum can happen on a positive-measured subset of responses. For the ReLU activation $\sigma(x) = \max(0, x)$, we completely classify cases where the ERM for a best two-layer neural network approximation attains its infimum. In recent applications of neural networks, where overfitting is commonplace, the failure to attain an infimum is avoided by ensuring that the system of equations $t_i = v_\theta(s_i)$, $i = 1, \dots, n$, has a solution. For a two-layer ReLU-activated network, we show when such a system of equations has a solution generically, i.e., when can such a neural network be fitted perfectly with probability one.

8.6 Applications

8.6.1 Performance evaluation of emergency call centers

Participants: Xavier Allamigeon, Marin Boyet, Baptiste Colin, Stéphane Gaubert.

Since 2014, we have been collaborating with Préfecture de Police (Régis Reboul and LcL Stéphane Raclot), more specifically with Brigade de Sapeurs de Pompiers de Paris (BSPP) and Direction de Sécurité de Proximité de l'agglomération parisienne (DSPAP), on the performance evaluation of the new organization (PFAU, "Plate forme d'appels d'urgence") to handle emergency calls to firemen and policemen in

the Paris area. We developed analytical models, based on Petri nets with priorities, and fluid limits, see [45], [46], [52]. In 2019, with four students of École polytechnique, Céline Moucer, Julia Escribe, Skandère Sahli and Alban Zammit, we performed case studies, showing the improvement brought by the two level filtering procedure.

Moreover, in 2019, this work has been extended to encompass the handling of health emergency calls, with a new collaboration, involving responsables from the four services of medical emergency aid of Assistance Publique – Hôpitaux de Paris (APHP), i.e., with SAMU75, 92, 93, 94, in the framework of a project coordinated by Dr. Christophe Leroy from APHP. As part of his PhD work, Marin Boyet have developed Petri net models capturing the characteristic of the centers (CRRA) handling emergency calls the SAMU, in order to make dimensioning recommendations. Following this, we have been strongly solicited by APHP during the pandemic of Covid-19 in order to determine crisis dimensioning of SAMU. Besides, we have initiated a new collaboration, with SAMU69, also on dimensioning.

In parallel, we have further investigated the theoretical properties of timed Petri nets with preselection and priority routing. We represent the behavior of these systems by piecewise affine dynamical systems. We use tools from the theory of nonexpansive mappings to analyze these systems. We establish an equivalence theorem between priority-free fluid timed Petri nets and semi-Markov decision processes, from which we derive the convergence to a periodic regime and the polynomial-time computability of the throughput. More generally, we develop an approach inspired by tropical geometry, characterizing the congestion phases as the cells of a polyhedral complex. These results are illustrated by the application to the performance evaluation of emergency call centers of SAMU in the Paris area. These results have been published in [15].

8.6.2 Optimal control of energy flexibilities in a stochastic environment

Participants: Maxime Grangereau, Stéphane Gaubert.

The PhD thesis of Maxime Grangereau [70] has been cosupervised by Emmanuel Gobet (CMAP), Stéphane Gaubert, and Wim van Aackooij (EDF Labs), it dealt with the application of stochastic control methods to the optimization of flexibilities in energy management.

A first series of results concern the application of mean-field control methods to the smart grid [27], including the modelling of storage resources and decentralized aspects [28]. Another contribution concerns a version of the Newton method in stochastic control [29]. A last series of work concern the multistage and stochastic extension of the Optimal Power Flow problem (OPF). We developed semidefinite relaxations, extending the ones which arise in static and deterministic OPF problems. We provided a priori conditions which guarantee the absence of relaxation gap, and also a posteriori methods allowing one to bound this relaxation gap. We applied this approach on examples of grids, with scenario trees representing the random solar power production [16].

8.6.3 Bilevel programming applied to optimal pricing of energy contracts

Stéphane Gaubert, Quentin Jacquet

The PhD thesis of Quentin Jacquet, is cosupervised by Stéphane Gaubert, Clémence Alasseur (EDF Labs), and Wim van Ackooij (EDF Labs). It concerns the application of bilevel programming methods to the pricing of electricity contracts. We investigated in [31] a new model of customer's response, based on a quadratic regularization. We showed that this model has qualitative properties and a realism similar to the classical models based on the logit-response, while being amenable to mathematical programming and polyhedral techniques, and so to exact solutions, via a reduction to quadratic complementary problems. An application to a set of instances representative of French electricity contracts was also developed [31].

8.6.4 POMDP and adaptive management

Participants: Marianne Akian.

With Luz Pascal (former internship, now LIS, Marseille) and Iadine Chades (CSIRO, Australia), we have developed an algorithm for Adaptive Management based on a hidden model Markov Decision Process (hmMDP) with a universal set of predefined models in [20]. hmMDP is a particular case of Partially observable MDP (POMDP). The main algorithm to solve such a problem is the SARSOP algorithm, which share some similarities with the combination of SDDP and tropical algorithms considered in [36]. In a work in progress, we are studying the convergence of SARSOP algorithm using the same techniques as in [36].

9 Bilateral contracts and grants with industry

9.1 Bilateral contracts with industry

Participants: Stéphane Gaubert.

- Stochastic optimization of multiple flexibilities and energies in micro-grids, collaboration with Wim Van Ackooij, from EDF labs, with the PhD work of Maxime Grangereau (CIFRE PhD), supervised by Emmanuel Gobet (CMAP) and cosupervised by Stéphane Gaubert.
- Optimal pricing of energy and services. Collaboration with Clémence Alasseur and Wim Van Ackooij, from EDF Labs, with the Phd Work of Quentin Jacquet (CIFRE PhD), supervised by Stéphane Gaubert.

10 Partnerships and cooperations

10.1 International initiatives

10.1.1 STIC/MATH/CLIMAT AmSud project

Participants: Marianne Akian, Stéphane Gaubert.

- Math AmSud Project ARGO, “Algebraic Real Geometry and Optimization” (2020-2021) between CMM (Chile), Univ. Buenos Aires (Argentina), Univ. Fed. Rio and Univ. Fed. Ceara (Brasil), Univ Savoie and CMAP, Ecole polytechnique (France), including in particular M. Akian and S. Gaubert of Tropical team.

10.1.2 Bilateral project FACCTS with the University of Chicago

Participants: Stéphane Gaubert.

- Bilateral project FACCTS, between the University of Chicago (Statistics) – Lek-Heng Lim– and Ecole polytechnique – Stéphane Gaubert– “Tropical geometry of deep learning”. This project was postponed owing to the Covid-19 crisis.

10.2 International research visitors

10.2.1 Visits to international teams

Participants: Xavier Allamigeon.

Research stays abroad

- X. Allamigeon: invitation to the “Discrete Optimization” thematic trimester at Hausdorff Institute for Mathematics, Sep.-Oct. 2021, Bonn, Germany.

10.3 National initiatives

10.3.1 ANR

Participants: Xavier Allamigeon.

- Project ANR JCJC CAPPs (“Combinatorial Analysis of Polytopes and Polyhedral Subdivisions”). Responsable: Arnau Padrol (IMJ-PRG, Sorbonne Université). Partners : IMJ-PRG (Sorbonne Université), INRIA Saclay (Tropical), LIGM (Université Paris-Est Marne-la-Vallée), LIF (Université Aix-Marseille), CERMICS (École Nationale des Ponts et Chaussées), LIX (École Polytechnique).

10.3.2 Programme Gaspard Monge pour l’optimisation, la recherche opérationnelle et leurs interactions avec les sciences des données

Participants: Xavier Allamigeon, Stéphane Gaubert.

- Méthodes tropicales pour le dimensionnement de centres d’appels : application à un centre de supervision EDF Participants : X. Allamigeon S. Gaubert, P. Bendotti (EDF) et T. Triboulet (EDF).

10.3.3 Centre des Hautes Études du Ministère de l’Intérieur

- Project “Optimisation de la performance de centres de traitement d’appels d’urgence en cas d’événements planifiés ou imprévus”, coordinated by X. Allamigeon, involving M. Boyet, B. Colin and S. Gaubert.

11 Dissemination

11.1 Promoting scientific activities

Participants: Marianne Akian, Xavier Allamigeon, Stéphane Gaubert, Yang Qi, Constantin Vernicos.

11.1.1 Scientific events: organisation

General chair, scientific chair

- S. Gaubert is the coordinator of the Gaspard Monge Program for Optimization, Operations Research and their interactions with data sciences (PGMO), a corporate sponsorship program, operated by Fondation Mathématique Jacques Hadamard, supported by Criteo, EDF, Orange and Thales, see [Pgmo site](#).

Member of the organizing committees

- X. Allamigeon: co-organizer of the workshop “**Tropical geometry and the geometry of linear programming**”, Hausdorff Institute for Mathematics, Sep. 2021, Bonn, Germany,
- S. Gaubert co-organizes the “**Séminaire Parisien d’Optimisation**” at Institut Henri Poincaré.
- S. Gaubert co-organizes the “**Séminaire Français d’Optimisation**” (online).
- C. Vernicos is in charge (“responsable thématique”) of the “Structures projectives” action within **GDR Platon**.
- C. Vernicos is coorganizer of the “Platon” monthly sessions of the French virtual seminar “**Groupes et géométries**”.

Organization of invited sessions

- Marianne Akian: organization with J. Darbon, P. Dower, and W. McEneaney of a session entitled “Optimal control and Games” at **SIAM CT’21** (SIAM Conference on Control and its Applications) July 2021, online.

11.1.2 Scientific events: selection

Chair of conference program committees

- S. Gaubert, co-chair of the **PGMO Days 2021, EDF Labs, 30 Nov – 1 Dec**.

Member of the conference program committees

- Marianne Akian: member of the scientific committee of “**Journées SMAI MODE 2022**”.
- Stéphane Gaubert
 - Organizer of the PGMO lecture of Sylvain Sorin, February 2021.

11.1.3 Journal

Member of the editorial boards

- Stéphane Gaubert: member of the editorial board of Journal of Dynamics and Games, Linear and Multilinear Algebra, RAIRO, Springer-SMAI book series.

11.1.4 Invited talks

- M. Akian:
 - MCA 2021 (Mathematical Congress of the Americas), July 12-16, 2021, online. Session “Mathematics of Planet Earth”, Talk: “Understanding and monitoring the evolution of the Covid-19 epidemic from medical emergency calls: the example of the Paris area”.
 - Online talk: “Tropical linear regression and mean payoff games: or, how to measure the distance to equilibria” at Workshop “**Tropical geometry and the geometry of linear programming**”, Hausdorff Institute for Mathematics, Sep. 2021, Bonn, Germany.
- S. Gaubert:
 - “Ambitropical convexity, mean payoff games and nonarchimedean convex programming”. **Combinatorics and Arithmetic for Physics: special days**, IHES, Dec. 2021.
 - Online talk: “Tropical convexity and its relation with mean payoff games and linear programming” at Workshop “**Tropical geometry and the geometry of linear programming**”, Hausdorff Institute for Mathematics, Sep. 2021, Bonn, Germany.

- Talk on “Ambitropical convexity, mean payoff games and nonarchimedean convex programming”, Combinatorics and Arithmetic for Physics: special days, IHES, Dec. 2021.
- C. Vernicos:
 - Talk for the "Séminaire virtuel francophone Groupes et Géométrie", 05/06/2021, "Entropie volumique des géométries de Hilbert et approximabilité".
 - Conference in Cortona, Italy, on "Gromov hyperbolicity and negative curvature in Complex Analysis" 09/6-09/10 2021. Talk on "Flag approximability and volume Entropy of Hilbert geometries".
- Y. Qi:
 - “On best approximations of high-dimensional semialgebraic data sets”, HCM Symposium, Hausdorff Center for Mathematics, August 26, 2021.
 - “Tropical linear regression and low-rank approximation – a first step in tropical data analysis”, TATERS, Boise State University, December 3, 2021.

11.1.5 Leadership within the scientific community

See Section 11.1.1.

11.1.6 Scientific expertise

- Marianne Akian: Scientific expertise for Datastorm and Air Liquide on the research and development works of Air Liquide in optimization of Air Separation Units (dec. 2020-jan. 2021).

11.1.7 Research administration

Inria research administration

- M. Akian: Alternate elected member of Inria’s Scientific Board.
- X. Allamigeon: member of the scientific committee of INRIA Saclay.

Other research administration

- M. Akian: member of the “Comité de Liaison” of SMAI MODE group, until May 2021.
- X. Allamigeon: elected member of the committee of applied mathematics department of Ecole Polytechnique.
- S. Gaubert, member of the “Commission d’appellation” of ENSTA.
- S. Gaubert, member of the committee of applied mathematics department of Ecole Polytechnique.
- C. Vernicos: Elected member and deputy-head rank B of CNU, section 25.
- C. Vernicos: Elected member of CVFU of University of Montpellier.

11.2 Teaching - Supervision - Juries

11.2.1 Teaching

- M. Akian
 - Course “Markov decision processes: dynamic programming and applications” joint between (3rd year of) ENSTA and M2 “Mathématiques et Applications”, U. Paris Saclay, “Optimization”, 30 hours.

- X. Allamigeon
 - Petites classes et encadrement d’enseignements d’approfondissement de Recherche Opérationnelle en troisième année à l’École Polytechnique (programme d’approfondissement de Mathématiques Appliquées) (niveau M1).
 - Cours du M2 “Optimisation” de l’Université Paris Saclay, cours partagé avec Céline Gicquel (LRI, Université Paris Sud).
 - Co-responsabilité du programme d’approfondissement en mathématiques appliquées (troisième année) à l’École Polytechnique.
- A. Bereau
 - Exercices classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- S. Gaubert
 - Co-head of the Master “Optimization” of University Paris-Saclay and IPP.
 - Course “Systèmes à Événements Discrets”, option MAREVA, ENSMP.
 - Course “Algèbre tropicale pour le contrôle optimal et les jeux” of “Contrôle, Optimisation et Calcul des Variations” (COCV) of M2 “Mathématiques et Applications” of Sorbonne University and École Polytechnique.
 - Lecture of Operations Research, third year of École Polytechnique. The lectures notes were published as a book [53].
- S. Liu
 - Exercices classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- O. Saadi
 - Exercices classes in the framework of a “Monitorat”.
- N. Vandame
 - Exercices classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.

11.2.2 Supervision

- PhD: Maxime Grangereau, registered at Univ. Paris Saclay, thesis supervisor: Emanuel Gobet, cosupervision: Stéphane Gaubert. The defense took place on March 23, 2021.
- PhD: Omar Saadi, registered at Univ. Paris Saclay thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian. The defense took place on December 17, 2021.
- PhD in progress: Marin Boyet, registered at Univ. Paris Saclay since October 2018, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.
- PhD in progress: Maël Forcier, registered at ENPC since September 2019, thesis supervisor: Vincent Leclère, cosupervision Stéphane Gaubert.
- PhD in progress: Quentin Canu, registered at Univ. Paris Saclay since October 2020, thesis supervisor: Georges Gonthier (INRIA), cosupervision: Xavier Allamigeon and Pierre-Yves Strub (LIX)
- PhD in progress: Shanqing Liu, registered at IPP (EDMH) since September 2020, thesis supervisor, M. Akian, co-supervised by S. Gaubert.

- PhD in progress: Quentin Jacquet, registered at IPP (EDMH) since November 2020, thesis supervisor, S. Gaubert, co-supervised by Clémence Alasseur and Wim van Ackooij.
- PhD in progress: Tom Ferragut, on the horosphérical products of Gromov-hyperbolic and Busemann convex metric spaces, registered at U. Montpellier since 2018, thesis supervisor: C. Vernicos, co-supervisor Jérémie Brieuessel.
- PhD in progress: Antoine Bereau, registered at IPP (EDMH) since September 2021, thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian.
- PhD in progress: Nicolas Vandame, registered at IPP (EDMH) since September 2021, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.

11.2.3 Juries

- M. Akian
 - Jury of the 2021 competition for a Maître de Conférence position in Applied Mathematics at Limoges University.
 - Jury of the 2021 competition for a Maître de Conférence position in Computer Science (Optimization) at Clermont University.
 - Jury of the PhD thesis of Dylan Dronnier (U. Paris Est, ENPC), Nov. 26, 2021.
 - Review and Jury of the PhD thesis of Adrien Lefranc (U. Paris Est, ENPC), Dec. 8, 2021.
 - Jury of the PhD thesis of Aurélien Desoeuvres (U. Montpellier), Dec. 9, 2021.
 - Jury of the PhD of Omar Saadi, Ecole polytechnique, Dec. 17, 2020.
- S. Gaubert
 - Jury of the PhD of Maxime Grangerau, Ecole polytechnique, March 23, 2021.
 - Jury of the PhD of Sebastian Tapia, Université de Bordeaux et Universida del Chile, November 8, 2021.
 - Jury of the PhD of Omar Saadi, Ecole polytechnique, Dec. 17, 2020.

11.3 Conferences, Seminars

- M. Akian
 - “10 ième Biennale Française des Mathématiques Appliquées et Industrielles” (SMAI), La Grande Motte, June 21-25, 2021. Session “Tests et traçage des contacts pour une surveillance épidémique efficace”, Talk “Probabilistic, mean-field and transport PDE models of Covid-19 epidemics, with variable contact rates and user mobility”.
 - SIAM Conference on Control and its Applications, July 19-21, 2021, online. Session “Optimal control and Games”, Talk: “The Reduction from Ergodic to Discounted Infinite Horizon Stochastic Zero-Sum Games Problems”.
 - Workshop Jeux Dynamiques, Quimper, Oct. 20-22, 2021. Talk: “Ambitropical convexity: The geometry of fixed point sets of Shapley operators”.
 - Probability seminar of IRMAR (Rennes, Nov. 2021). Talk: “Modèles probabilistes, champ moyen, EDP de transport pour l’épidémie de Covid-19 avec taux de contact variables”.
- X. Allamigeon
 - CERMICS seminar, Ecole des Ponts Paristech, January 2021. Title: “What tropical geometry tells us about the complexity of linear programming”.
 - Colloquium “Facets of Complexity” (joint to Freie Universität Berlin, Technische Universität Berlin, and Humboldt-Universität zu Berlin), June 2021. Title: “Formalizing the theory of polyhedra in a proof assistant”.

- SIAM Conference on Applied Algebraic Geometry (AG21), invited session on “Applications of Tropical Geometry”, August 16 - 20, 2021. Title: “Tropical Nash Equilibria and Complementarity Problems”.
- Seminar of the thematic trimester “Discrete Optimization”, Hausdorff Research Institute for Mathematics, Bonn, October 2021. Title: “What tropical geometry tells us about the complexity of linear programming”.
- M. Forcier, Talk: Multistage Stochastic Linear Programming and Polyhedral Geometry, PGMO Days, Nov 30.-Dec. 1, 2021.
- S. Gaubert
 - SIAM Conference on Control and its Applications, July 19-21, 2021, online. Session “Optimal control and Games”, Talk: “Ambitropical convexity”.
 - SIAM Conference on Applied Algebraic Geometry (AG21), invited session on “Applications of Tropical Geometry”, August 16 - 20, 2021. Title: “Ambitropical convexity”.
 - Workshop Jeux Dynamiques, Quimper, Oct. 20-22, 2021. Talk: “Tropical linear regression and mean payoff games”.
 - Seminar of Algebra, Bar Ilan University (on-line), The tropicalization of non-archimedean convex semialgebraic sets and its relation with mean payoff games, April 07, 2021.
 - Séminaire parisien de théorie des jeux, IHP, January 2021: “Ambitropical convexity, or the geometry of fixed-point sets of Shapley operators”.
- A. Gutiérrez: Poster presentation at the conference "Frontiers of operator theory" at CIRM Marseille Luminy, November 29 - December 3, 2021.
- S. Liu: PGMO Days, Nov 30.-Dec. 1, 2021. Title: “A multilevel fast-marching method”.
- Y. Qi
 - Workshop III: Mathematical Foundations and Algorithms for Tensor Computations, part of the long program Tensor Methods and Emerging Applications to the Physical and Data Sciences, Institute for Pure and Applied Mathematics, University of California, Los Angeles, May 3-6, 2021, Online. Title: “Inner and outer approximations of tropical polytopes and their applications in tropical tensors”.
 - “Geometric techniques for tensor computations”, Séminaire du CMAP, École Polytechnique, March 2, 2021.
 - PGMO Days, Nov 30.-Dec. 1, 2021. Title: “Tropical linear regression and low-rank approximation — a first step in tropical data analysis”.
- O. Saadi
 - ROADEF April 26-30, 2021, Online. Title: “Tropical linear regression and its applications to markets with repeated invitations to tender”.
 - CJC-MA 2021 (Congrès des Jeunes Chercheuses et Chercheurs en Mathématiques Appliquées), Oct. 27-29, 2021, Ecole polytechnique. Title: “Tropical linear regression and mean payoff games: or, how to measure the distance to equilibria”.

12 Scientific production

12.1 Major publications

- [1] M. Akian, S. Gaubert and A. Guterman. ‘Tropical polyhedra are equivalent to mean payoff games’. In: *Internat. J. Algebra Comput.* 22.1 (2012), pp. 1250001, 43. DOI: [10.1142/S0218196711006674](https://doi.org/10.1142/S0218196711006674). eprint: [0912.2462](https://arxiv.org/abs/0912.2462). URL: <http://dx.doi.org/10.1142/S0218196711006674>.

- [2] M. Akian, S. Gaubert and R. Nussbaum. ‘Uniqueness of the fixed point of nonexpansive semidifferentiable maps’. In: *Transactions of the American Mathematical Society* 368.2 (Feb. 2016). Also arXiv:1201.1536. DOI: [10.1090/S0002-9947-2015-06413-7](https://doi.org/10.1090/S0002-9947-2015-06413-7). URL: <https://hal.inria.fr/hal-00783682>.
- [3] M. Akian, S. Gaubert and R. Bapat. ‘Non-archimedean valuations of eigenvalues of matrix polynomials’. In: *Linear Algebra and its Applications* 498 (June 2016). Also arXiv:1601.00438, pp. 592–627. DOI: [10.1016/j.laa.2016.02.036](https://doi.org/10.1016/j.laa.2016.02.036). URL: <https://hal.inria.fr/hal-01251803>.
- [4] M. Akian, S. Gaubert and C. Walsh. ‘The max-plus Martin boundary’. In: *Doc. Math.* 14 (2009), pp. 195–240.
- [5] X. Allamigeon, P. Benchimol, S. Gaubert and M. Joswig. ‘Combinatorial simplex algorithms can solve mean payoff games’. In: *SIAM J. Opt.* 24.4 (2015), pp. 2096–2117. eprint: [1309.5925](https://arxiv.org/abs/1309.5925).
- [6] X. Allamigeon, S. Gaubert, E. Goubault, S. Putot and N. Stott. ‘A scalable algebraic method to infer quadratic invariants of switched systems’. In: *Proceedings of the International Conference on Embedded Software (EMSOFT)*. Best paper award. The extended version of this conference article appeared in *ACM Trans. Embed. Comput. Syst.*, 15(4):69:1–69:20, September 2016. 2015.
- [7] X. Allamigeon, P. Benchimol, S. Gaubert and M. Joswig. ‘Log-barrier interior point methods are not strongly polynomial’. In: *SIAM Journal on Applied Algebra and Geometry* 2.1 (2018). <https://arxiv.org/abs/1708.01544> - This paper supersedes arXiv:1405.4161. 31 pages, 5 figures, 1 table, pp. 140–178. DOI: [10.1137/17M1142132](https://doi.org/10.1137/17M1142132). URL: <https://hal.inria.fr/hal-01674959>.
- [8] J. Bolte, S. Gaubert and G. Vigerál. ‘Definable zero-sum stochastic games’. In: *Mathematics of Operations Research* 40.1 (2014). Also arxiv:1301.1967, pp. 171–191. DOI: [10.1287/moor.2014.0666](https://doi.org/10.1287/moor.2014.0666). URL: <http://dx.doi.org/10.1287/moor.2014.0666>.
- [9] S. Friedland, S. Gaubert and L. Han. ‘Perron-Frobenius theorem for nonnegative multilinear forms and extensions’. In: *Linear Algebra and its Applications* 438.2 (2013). This paper was included in a list of “10 Notable Papers from the journal *Linear Algebra & Its Applications* over the last 50 years” at the occasion of the <https://www.journals.elsevier.com/linear-algebra-and-its-applications/10-notable-papers-linear-algebra-applications-50-years> golden anniversary of the journal, celebrated in 2018., pp. 738–749. eprint: [0905.1626](https://arxiv.org/abs/0905.1626). URL: <http://dx.doi.org/10.1016/j.laa.2011.02.042>.
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- [12] C. Walsh. ‘The horofunction boundary and isometry group of the Hilbert geometry’. In: *Handbook of Hilbert Geometry*. Vol. 22. IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society, 2014. URL: <https://hal.inria.fr/hal-00782827>.

12.2 Publications of the year

International journals

- [13] M. Akian, S. Gaubert, Z. Qu and O. Saadi. ‘Multiply Accelerated Value Iteration for Non-Symmetric Affine Fixed Point Problems and application to Markov Decision Processes’. In: *SIAM Journal on Matrix Analysis and Applications* (2021). URL: <https://hal.inria.fr/hal-03059718>.
- [14] X. Allamigeon, P. Benchimol, S. Gaubert and M. Joswig. ‘What Tropical Geometry Tells Us about the Complexity of Linear Programming’. In: *SIAM Review* 63.1 (4th Feb. 2021), pp. 123–164. DOI: [10.1137/20M1380211](https://doi.org/10.1137/20M1380211). URL: <https://hal.inria.fr/hal-03505719>.
- [15] X. Allamigeon, M. Boyet and S. Gaubert. ‘Piecewise Affine Dynamical Models of Timed Petri Nets – Application to Emergency Call Centers’. In: *Fundamenta Informaticae* 183.3-4 (2021), pp. 169–201. URL: <https://hal.archives-ouvertes.fr/hal-02550006>.

- [16] M. Grangereau, W. Van Ackooij and S. Gaubert. ‘Multi-stage Stochastic Alternating Current Optimal Power Flow with Storage: Bounding the Relaxation Gap’. In: *Electric Power Systems Research* 206 (11th Jan. 2022), p. 107774. DOI: [10.1016/j.epsr.2022.107774](https://doi.org/10.1016/j.epsr.2022.107774). URL: <https://hal.archives-ouvertes.fr/hal-03236238>.
- [17] A. W. Gutiérrez and A. Karlsson. ‘Comments on the cosmic convergence of nonexpansive maps’. In: *Journal of Fixed Point Theory and Applications* 23.4 (Nov. 2021). DOI: [10.1007/s11784-021-00896-8](https://doi.org/10.1007/s11784-021-00896-8). URL: <https://hal.archives-ouvertes.fr/hal-03426011>.
- [18] L.-H. Lim, M. Michalek and Y. Qi. ‘Best k -layer neural network approximations’. In: *Constructive Approximation* (7th June 2021). URL: <https://hal.inria.fr/hal-03088287>.
- [19] C. Vernicos and C. Walsh. ‘Flag-approximability of convex bodies and volume growth of Hilbert geometries’. In: *Annales Scientifiques de l’École Normale Supérieure* 54 (2021), pp. 1297–1315. URL: <https://hal.archives-ouvertes.fr/hal-01423693>.

International peer-reviewed conferences

- [20] L. V. Pascal, M. Akian, S. Nicol and I. Chades. ‘A Universal 2-state n-action Adaptive Management Solver’. In: Proceedings of the AAAI Conference on Artificial Intelligence. Vol. 35. 17. virtual conference, United States, 18th May 2021, pp. 14884–14892. URL: <https://hal.inria.fr/hal-03505465>.

Conferences without proceedings

- [21] R. Affeldt, X. Allamigeon, Y. Bertot, Q. Canu, C. Cohen, P. Roux, K. Sakaguchi, E. Tassi, L. Théry and A. Trunov. ‘Porting the Mathematical Components library to Hierarchy Builder’. In: the COQ Workshop 2021. virtuel- Rome, Italy, 2nd July 2021. URL: <https://hal.archives-ouvertes.fr/hal-03463762>.

Reports & preprints

- [22] M. Akian, S. Gaubert and S. Vannucci. *Ambitropical convexity: The geometry of fixed point sets of Shapley operators*. 17th Aug. 2021. URL: <https://hal.inria.fr/hal-03504873>.
- [23] M. Akian, S. Gaubert, Y. Qi and O. Saadi. *Tropical linear regression and mean payoff games: or, how to measure the distance to equilibria*. 21st June 2021. URL: <https://hal.inria.fr/hal-03504875>.
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- [25] M. Forcier, S. Gaubert and V. Leclère. *Exact quantization of multistage stochastic linear problems*. 20th July 2021. URL: <https://hal.inria.fr/hal-03504876>.
- [26] M. Forcier and V. Leclère. *Generalized adaptive partition-based method for two-stage stochastic linear programs : convergence and generalization*. 25th Jan. 2022. URL: <https://hal-enpc.archives-ouvertes.fr/hal-03542218>.
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- [29] E. GOBET and M. Grangereau. *Newton method for stochastic control problems*. 13th Jan. 2021. URL: <https://hal.archives-ouvertes.fr/hal-03108627>.
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12.3 Cited publications

- [32] X. Allamigeon, S. Gaubert and M. Skomra. ‘Tropical spectrahedra’. In: *Discrete and Computational Geometry* 63 (Feb. 2020), pp. 507–548. DOI: [10.1007/s00454-020-00176-1](https://doi.org/10.1007/s00454-020-00176-1). URL: <https://hal.inria.fr/hal-01422639>.
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