RESEARCH CENTRE

**Inria Saclay Center at Institut Polytechnique de Paris**

2022
ACTIVITY REPORT

Project-Team

# TROPICAL

## Tropical methods: structures, algorithms and interactions

IN COLLABORATION WITH: Centre de Mathématiques Appliquées (CMAP)

**DOMAIN**

**Applied Mathematics, Computation and Simulation**

**THEME**

**Optimization and control of dynamic systems**

Inria

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# Project-Team TROPICAL

*Creation of the Project-Team: 2018 July 01*


## Keywords

### Computer sciences and digital sciences

A1.2.4. – QoS, performance evaluation

A2.3.3. – Real-time systems

A2.4. – Formal method for verification, reliability, certification

A6.2.5. – Numerical Linear Algebra

A6.2.6. – Optimization

A6.4.2. – Stochastic control

A6.4.6. – Optimal control

A7.2.4. – Mechanized Formalization of Mathematics

A8.1. – Discrete mathematics, combinatorics

A8.3. – Geometry, Topology

A8.9. – Performance evaluation

A8.11. – Game Theory

A9.6. – Decision support

### Other research topics and application domains

B4.3. – Renewable energy production

B4.4. – Energy delivery

B4.4.1. – Smart grids

B6.6. – Embedded systems

B8.4. – Security and personal assistance

B8.4.1. – Crisis management

# 1 Team members, visitors, external collaborators

### Research Scientists

- Stephane Gaubert [Team leader, INRIA, Senior Researcher, HDR]
- Marianne Akian [INRIA, Senior Researcher, HDR]
- Xavier Allamigeon [INRIA, Researcher]
- Gleb Koshevoy [Institute of Information Transmission Problems (Kharkevich Institute), Russian Academy of Sciences, Senior Researcher, from Nov 2022]
- Yang Qi [INRIA, Starting Research Position]
- Cormac Walsh [INRIA, Researcher]

### Faculty Member

- Constantin Vernicos [Univ Montpellier, Associate Professor, until Aug 2022, In delegation, HDR]

### Post-Doctoral Fellow

- Armando Gutiérrez [Académie de Finlande, until Jul 2022]

### PhD Students

- Amanda Bigel [INRIA, from Oct 2022]
- Antoine Béreau [Ecole polytechnique (PhD contract funded by ENS Rennes)]
- Quentin Canu [ENS PARIS-SACLAY]
- Quentin Jacquet [EDF]
- Shanqing Liu [ECOLE POLY PALAISEAU]
- Nicolas Vandame [ECOLE POLY PALAISEAU]

### Interns and Apprentices

- Amanda Bigel [INRIA, from May 2022 until Sep 2022]
- Jiawei He [INRIA, from Apr 2022 until Jul 2022]

### Administrative Assistant

- Hanadi Dib [INRIA]

# 2 Overall objectives

The project develops tropical methods motivated by applications arising in decision theory (deterministic and stochastic optimal control, game theory, optimization and operations research), in the analysis or control of classes of dynamical systems (including timed discrete event systems and positive systems), in the verification of programs and systems, and in the development of numerical algorithms. Tropical algebra tools are used in interaction with various methods, coming from convex analysis, Hamilton–Jacobi partial differential equations, metric geometry, Perron-Frobenius and nonlinear fixed-point theories, combinatorics or algorithmic complexity. The emphasis of the project is on mathematical modelling and computational aspects.

The subtitle of the Tropical project, namely, “structures, algorithms, and interactions”, refers to the spirit of our research, including a methodological component, computational aspects, and finally interactions with other scientific fields or real world applications, in particular through mathematical modelling.

### 2.1 Scientific context

Tropical algebra, geometry, and analysis have enjoyed spectacular development in recent years. Tropical structures initially arose to solve problems in performance evaluation of discrete event systems [59], combinatorial optimization [66], or automata theory [102]. They also arose in mathematical physics and asymptotic analysis [92, 88]. More recently, these structures have appeared in several areas of pure mathematics, in particular in the study of combinatorial aspects of algebraic geometry [82, 117, 107, 87], in algebraic combinatorics [76], and in arithmetics [70]. Also, further applications of tropical methods have appeared, including optimal control [93], program invariant computation [52] and timed systems verification [91], and zero-sum games [1].

The term ‘tropical’ generally refers to algebraic structures in which the laws originate from optimization processes. The prototypical tropical structure is the max-plus semifield, consisting of the real numbers, equipped with the maximum, thought of as an additive law, and the addition, thought of as a multiplicative law. Tropical objects appear as limits of classical objects along certain deformations (“log-limits sets” of Bergman, “Maslov dequantization”, or “Viro deformation”). For this reason, the introduction of tropical tools often yields new insights into old familiar problems, leading either to counterexamples or to new methods and results; see for instance [117, 97]. In some applications, like optimal control, discrete event systems, or static analysis of programs, tropical objects do not appear through a limit procedure, but more directly as a modelling or computation/analysis tool; see for instance [113, 59, 85, 67].

Tropical methods are linked to the fields of positive systems and of metric geometry [99], [12]. Indeed, tropically linear maps are monotone (a.k.a. order-preserving). They are also nonexpansive in certain natural metrics (sup-norm, Hopf oscillation, Hilbert’s projective metric, . . . ). In this way, tropical dynamical systems appear to be special cases of nonexpansive, positive, or monotone dynamical systems, which are studied as part of linear and non-linear Perron-Frobenius theory [90], [2]. Such dynamical systems are of fundamental importance in the study of repeated games [96]. Monotonicity properties are also essential in the understanding of the fixed points problems which determine program invariants by abstract interpretation [71]. The latter problems are actually somehow similar to the ones arising in the study of zero-sum games; see [6]. Moreover, positivity or monotonicity methods are useful in population dynamics, either in a discrete space setting [114] or in a PDE setting [61]. In such cases, solving tropical problems often leads to solutions or combinatorial insights on classical problems involving positivity conditions (e.g., finding equilibria of dynamical systems with nonnegative coordinates, understanding the qualitative and quantitative behavior of growth rates / Floquet eigenvalues [10], etc). Other applications of Perron-Frobenius theory originate from quantum information and control [106, 111].

## 3 Research program

### 3.1 Optimal control and zero-sum games

The dynamic programming approach allows one to analyze one or two-player dynamic decision problems by means of operators, or partial differential equations (Hamilton–Jacobi or Isaacs PDEs), describing the time evolution of the value function, i.e., of the optimal reward of one player, thought of as a function of the initial state and of the horizon. We work especially with problems having long or infinite horizon, modelled by stopping problems, or ergodic problems in which one optimizes a mean payoff per time unit. The determination of optimal strategies reduces to solving nonlinear fixed point equations, which are obtained either directly from discrete models, or after a discretization of a PDE.

**The geometry of solutions of optimal control and game problems** Basic questions include, especially for stationary or ergodic problems, the understanding of existence and uniqueness conditions for the solutions of dynamic programming equations, for instance in terms of controllability or ergodicity properties, and more generally the understanding of the structure of the full set of solutions of stationary Hamilton–Jacobi PDEs and of the set of optimal strategies. These issues are already challenging in the one-player deterministic case, which is an application of choice of tropical methods, since the Lax-Oleinik semigroup, i.e., the evolution semigroup of the Hamilton-Jacobi PDE, is a linear operator in the tropical sense. Recent progress in the deterministic case has been made by combining dynamical systems and PDE techniques (weak KAM theory [73]), and also using metric geometry ideas (abstract boundaries can be used to represent the sets of solutions [86], [4]). The two player case is challenging, owing to the lack of compactness of the analogue of the Lax-Oleinik semigroup and to a richer geometry. The conditions of solvability of ergodic problems for games (for instance, solvability of ergodic Isaacs PDEs), and the representation of solutions are only understood in special cases, for instance in the finite state space case, through tropical geometry and non-linear Perron-Frobenius methods [49], [50], [2].

**Algorithmic aspects: from combinatorial algorithms to the attenuation of the curse of dimensionality** Our general goal is to push the limits of solvable models by means of fast algorithms adapted to large scale instances. Such instances arise from discrete problems, in which the state space may so large that it is only accessible through local oracles (for instance, in some web ranking applications, the number of states may be the number of web pages) [74]. They also arise from the discretization of PDEs, in which the number of states grows exponentially with the number of degrees of freedom, according to the "curse of dimensionality". A first line of research is the development of *new approximation methods for the value function*. So far, classical approximations by linear combinations have been used, as well as approximation by suprema of linear or quadratic forms, which have been introduced in the setting of dual dynamic programming and of the so called "max-plus basis methods" [75]. We believe that more concise or more accurate approximations may be obtained by unifying these methods. Also, some max-plus basis methods have been shown to *attenuate the curse of dimensionality* for very special problems (for instance involving switching) [94, 79]. This suggests that the complexity of control or games problems may be measured by more subtle quantities that the mere number of states, for instance, by some forms of metric entropy (for example, certain large scale problems have a low complexity owing to the presence of decomposition properties, "highway hierarchies", etc.). A second line of of our research is the development of *combinatorial algorithms*, to solve large scale zero-sum two-player problems with discrete state space. This is related to current open problems in algorithmic game theory. In particular, the existence of polynomial-time algorithms for games with ergodic payment is an open question. See e.g. [53] for a polynomial time average complexity result derived by tropical methods. The two lines of research are related, as the understanding of the geometry of solutions allows to develop better approximation or combinatorial algorithms.

## 3.2 Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

Several applications (including population dynamics [10] and discrete event systems [59, 69, 55]) lead to studying classes of dynamical systems with remarkable properties: preserving a cone, preserving an order, or being nonexpansive in a metric. These can be studied by techniques of non-linear Perron-Frobenius theory [2] or metric geometry [11]. Basic issues concern the existence and computation of the "escape rate" (which determines the throughput, the growth rate of the population), the characterizations of stationary regimes (non-linear fixed points), or the study of the dynamical properties (convergence to periodic orbits). Nonexpansive mappings also play a key role in the "operator approach" to zero-sum games, since the one-day operators of games are nonexpansive in several metrics, see [8].

## 3.3 Tropical algebra and convex geometry

The different applications mentioned in the other sections lead us to develop some basic research on tropical algebraic structures and in convex and discrete geometry, looking at objects or problems with a "piecewise-linear " structure. These include the geometry and algorithmics of tropical convex sets [58],[1], tropical semialgebraic sets [34], the study of semi-modules (analogues of vector spaces when the base field is replaced by a semi-field), the study of systems of equations linear in the tropical sense, investigating for instance the analogues of the notions of rank, the analogue of the eigenproblems [51], and more generally of systems of tropical polynomial equations. Our research also builds on, and concern, classical convex and discrete geometry methods.

### 3.4 Tropical methods applied to optimization, perturbation theory and matrix analysis

Tropical algebraic objects appear as a deformation of classical objects thought various asymptotic procedures. A familiar example is the rule of asymptotic calculus,

$$e^{-a/\epsilon} + e^{-b/\epsilon} \asymp e^{-\min(a,b)/\epsilon} \quad , \qquad e^{-a/\epsilon} \times e^{-b/\epsilon} = e^{-(a+b)/\epsilon} \quad , \tag{1}$$

when $\epsilon \to 0^+$. Deformations of this kind have been studied in different contexts: large deviations, zero-temperature limits, Maslov's "dequantization method" [92], non-archimedean valuations, log-limit sets and Viro's patchworking method [117], etc.

This entails a relation between classical algorithmic problems and tropical algorithmic problems, one may first solve the $\epsilon = 0$ case (non-archimedean problem), which is sometimes easier, and then use the information gotten in this way to solve the $\epsilon = 1$ (archimedean) case.

In particular, tropicalization establishes a connection between polynomial systems and piecewise affine systems that are somehow similar to the ones arising in game problems. It allows one to transfer results from the world of combinatorics to "classical" equations solving. We investigate the consequences of this correspondence on complexity and numerical issues. For instance, combinatorial problems can be solved in a robust way. Hence, situations in which the tropicalization is faithful lead to improved algorithms for classical problems. In particular, scalings for the polynomial eigenproblems based on tropical preprocessings have started to be used in matrix analysis [80, 84].

Moreover, the tropical approach has been recently applied to construct examples of linear programs in which the central path has an unexpectedly high total curvature [54],[7], and it has also led to positive polynomial-time average case results concerning the complexity of mean payoff games. Similarly, we are studying semidefinite programming over non-archimedean fields [34], [57], with the goal to better understand complexity issues in classical semidefinite and semi-algebraic programming.

## 4 Application domains

### 4.1 Discrete event systems (manufacturing systems, networks, emergency call centers)

One important class of applications of max-plus algebra comes from discrete event dynamical systems [59]. In particular, modelling timed systems subject to synchronization and concurrency phenomena leads to studying dynamical systems that are non-smooth, but which have remarkable structural properties (nonexpansiveness in certain metrics , monotonicity) or combinatorial properties. Algebraic methods allow one to obtain analytical expressions for performance measures (throughput, waiting time, etc). A recent application, to emergency call centers, can be found in [55].

### 4.2 Optimal control and games

Optimal control and game theory have numerous well established applications fields: mathematical economy and finance, stock optimization, optimization of networks, decision making, etc. In most of these applications, one needs either to derive analytical or qualitative properties of solutions, or design exact or approximation algorithms adapted to large scale problems.

### 4.3 Operations Research

We develop, or have developed, several aspects of operations research, including the application of stochastic control to optimal pricing, optimal measurement in networks [108]. Applications of tropical methods arise in particular from discrete optimization [67], [68], scheduling problems with and-or constraints [98], or product mix auctions [116].

### 4.4 Computing program and dynamical systems invariants

A number of programs and systems verification questions, in which safety considerations are involved, reduce to computing invariant subsets of dynamical systems. This approach appears in various guises in computer science, for instance in static analysis of program by abstract interpretation, along the lines of P. and R. Cousot [71], but also in control (eg, computing safety regions by solving Isaacs PDEs). These invariant sets are often sought in some tractable effective class: ellipsoids, polyhedra, parametric classes of polyhedra with a controlled complexity (the so called "templates" introduced by Sankaranarayanan, Sipma and Manna [110]), shadows of sets represented by linear matrix inequalities, disjunctive constraints represented by tropical polyhedra [52], etc. The computation of invariants boils down to solving large scale fixed point problems. The latter are of the same nature as the ones encountered in the theory of zero-sum games, and so, the techniques developed in the previous research directions (especially methods of monotonicity, nonexpansiveness, discretization of PDEs, etc) apply to the present setting, see e.g. [78, 81] for the application of policy iteration type algorithms, or for the application for fixed point problems over the space of quadratic forms [6]. The problem of computation of invariants is indeed a key issue needing the methods of several fields: convex and nonconvex programming, semidefinite programming and symbolic computation (to handle semialgebraic invariants), nonlinear fixed point theory, approximation theory, tropical methods (to handle disjunctions), and formal proof (to certify numerical invariants or inequalities).

# 5 Social and environmental responsibility

### 5.1 Impact of research results

The team has developed collaborations on the dimensioning of emergency call centers, with Préfecture de Police (Plate Forme d'Appels d'Urgence - PFAU - 17-18-112, operated jointly by Brigade de sapeurs pompiers de Paris and by Direction de la sécurité de proximité de l'agglomération parisienne) and also with the Emergency medical services of Assistance Publique – Hôpitaux de Paris (Centre 15 of SAMU75, 92, 93 and 94). This work is described further in Section 8.6.1. A recent extension of this work deals with the modelling of medical emergency services, with the project "URGE" which started at the fall 2022, in the framework of the joint INRIA & AP-HP "Bernoulli" lab.

# 6 Highlights of the year

### 6.1 Awards

- X. Allamigeon received the "Prix Inria – Académie des sciences jeunes chercheurs et jeunes chercheuses", see the information on the INRIA site.

# 7 New software and platforms

### 7.1 New software

#### 7.1.1 Coq-Polyhedra

**Name:** Coq-Polyhedra

**Keywords:** Coq, Polyhedra, Automated theorem proving, Linear optimization

**Scientific Description:** Coq-Polyhedra is a library providing a formalization of convex polyhedra in the Coq proof assistant. While still in active development, it provides an implementation of the simplex method, and already handles the basic properties of polyhedra such as emptiness, boundedness, membership. Several fundamental results in the theory of convex polyhedra, such as Farkas Lemma, duality theorem of linear programming, and Minkowski Theorem, are also formally proved.

The formalization is based on the Mathematical Components library, and makes an extensive use of the boolean reflection methodology.

**Functional Description:** Coq-Polyhedra is a library which aims at formalizing convex polyhedra in Coq

**News of the Year:** Coq-Polyhedra now provides most of the basic operations on polyhedra. They are expressed on a quotient type that avoids reasoning with particular inequality representations. They include : * the construction of elementary polyhedra (half-spaces, hyperplanes, affine spaces, orthants, simplices, etc) * basic operations such as intersection, projection (thanks to the formalization of the Fourier-Motzkin algorithm), image under linear functions, computations of convex hulls, finitely generated cones, etc. * computation of affine hulls of polyhedra, as well as their dimension

Thanks to this, we have made huge progress on the formalization of the combinatorics of polyhedra. The poset of faces, as well as its fundamental properties (lattice, gradedness, atomicity and co-atomicity, etc) are now formalized. The manipulation of the faces is based on an extensive use of canonical structures, that allows to get the most appropriate inequality representations for reasoning. In this way, we arrive at very concise and elegant proofs, closer to the pen-and-paper ones.

**URL:** `https://github.com/nhojem/Coq-Polyhedra`

**Publications:** hal-01673390, hal-03151656, hal-03915661, hal-01967575, hal-01967576

**Contact:** Xavier Allamigeon

**Participants:** Xavier Allamigeon, Vasileios Charisopoulos, Quentin Canu, Ricardo Katz, Pierre-Yves Strub

**Partners:** CIFASIS, Ecole Polytechnique

### 7.1.2 EmergencyEval

**Keywords:** Dynamic Analysis, Simulation, Ocaml, Emergency, Firefighters, Police

**Scientific Description:** This software aims at enabling the definition of a Petri network execution semantic, as well as the instanciation and execution of said network using the aforedefined semantic.

The heart of the project dwells in its kernel which operates the step-by-step execution of the network, obeying rules provided by an oracle. This user-defined and separated oracle computes the information necessary to the kernel for building the next state using the current state. The base of our software is the framework for the instanciation and execution of Petri nets, without making assumptions regarding the semantic.

In the context of the study of the dynamics of emergency call centers, a second part of this software is the definition and implementation of the semantic of call centers modelized as Petri nets, and more specifically timed prioritized Petri nets. A module interoperating with the kernel enables to include all the operational specificities of call centers (urgency level, discriminating between operators and callers ...) while guaranteeing the genericity of the kernal which embeds the Petri net formalism as such.

**Functional Description:** In order to enable the quantitative study of the throughput of calls managed by emergency center calls and the assesment of various organisationnal configurations considered by the stakeholders (firefighters, police, medical emergency service of the 75, 92, 93 and 94 French departments), this software modelizes their behaviours by resorting to extensions of the Petri net formalism. Given a call transfer protocol in a call center, which corresponds to a topology and an execution semantic of a Petri net, the software generates a set of entering calls in accord with the empirically observed statistic ditributions (share of very urgent calls, conversation length), then simulates its management by the operators with respect to the defined protocol. Transitional regimes phenomenons (peak load, support) which are not yet handled by mathematical analysis could therefore be studied. The ouput of the software is a log file which is an execution trace of the simulation featuring extensive information in order to enable the analysis of the data for providing simulation-based insights for decision makers.

The software relies on a Petri net simulation kernel designed to be as modular and adaptable as possible, fit for simulating other Petri-net related phenomenons, even if their semantic differ greatly.

**Contact:** Xavier Allamigeon

**Participants:** Xavier Allamigeon, Benjamin Nguyen-Van-Yen

# 8 New results

## 8.1 Optimal control and zero-sum games

### 8.1.1 Multiply Accelerated Value Iteration Algorithms For Classes of Markov Decision Processes

**Participants:** Marianne Akian, Stéphane Gaubert, Omar Saadi.

Accelerated gradient algorithms in convex optimization were introduced by Nesterov. A fundamental question is whether similar acceleration schemes work for the iteration of nonexpansive mappings. In a joint work with Zheng Qu (Hong Kong University) [13], motivated by the analysis of Markov decision processes and zero-sum repeated games, we study fixed point problems for Shapley operators, i.e., for sup-norm nonexpansive and order preserving mapping. We deal more especially with affine operators, corresponding to zero-player problems – the latter can be used as a building block for one or two player problems, by means of policy iteration. For an affine operator, associated to a Markov chain, the acceleration property can be formalized as follows: one should replace an original scheme with a convergence rate $1-\Theta(\epsilon)$ by a convergence rate $1-\Theta(\epsilon^{1/2})$ where $\epsilon$ is the spectral gap of the Markov chain. We characterize the spectra of Markov chains for which this acceleration is possible. We also characterize the spectra for which a multiple acceleration is possible, leading to a rate of $1-\Theta(\epsilon^{1/d})$ for $d > 2$.

### 8.1.2 Polyhedral representation of multi-stage stochastic linear problems

**Participants:** Maël Forcier, Stéphane Gaubert.

In [43] (joint work with Vincent Leclère, ENPC), we study multistage stochastic problems with a linear structure and general cost distribution. We obtain an exact quantization result, showing that a multistage problem with a continuous cost distribution is equivalent to a problem with a discrete distribution, constructed by exploiting results from polyhedral geometry. In particular, we introduced a generalization of fiber polytopes, which characterize the value function. We deduce polynomial-time solvability results in fixed dimension, both for exact and approximated versions of the problem. Further results are presented in the PhD thesis [77].

### 8.1.3 Tropical numerical methods for stochastic control problems

**Participants:** Marianne Akian.

With Benoît Tran (FGV EMAp, Brazil), and Jean-Philippe Chancelier (CERMICS, ENPC), we were interested in the numerical solution of the dynamic programming equation of discrete time stochastic control problems, and developped and studied in [115] several algorithms combining the tropical or the max-plus based numerical method of McEneaney [95, 93], the stochastic max-plus scheme proposed by Zheng Qu [105], and the stochastic dual dynamic programming (SDDP) algorithm of Pereira and Pinto [100], In particular, in [38, 19], we considered and studied a general algorithm inspired by both tropical numerical methods and SDDP algorithm and which can be seen as a generalization of the algorithms proposed in [101, 62]).

In a work presented in [19], we also show that in the case of the dynamic programming equation associated to a partially observable Markov Decision Process (POMDP), it is similar to the so called point based algorithms developped in [103, 89, 112], which includes in particular SARSOP algorithm.

With Luz Pascal (QUT & CSIRO, Australia), we are also studying the convergence of SARSOP algorithm using the same techniques as in [38].

### 8.1.4 Highway hierarchies for Hamilton-Jacobi-Bellman (HJB) PDEs

**Participants:** Marianne Akian, Stéphane Gaubert, Shanqing Liu.

Hamilton-Jacobi-Bellman equations arise as the dynamic programming equations of deterministic or stochastic optimal control problems. They allow to obtain the global optimum of these problems and to synthetize an optimal feedback control, leading to a solution robust against system perturbations. Several methods have been proposed in the litterature to bypass the obstruction of curse of dimensionality of such equations, assuming a certain structure of the problem, and/or using "unstructured discretizations", that are not based on given grids. Among them, one may cite tropical numerical method, and probabilistic numerical method. On another direction, "highway hierarchies", developped by Sanders, Schultes and coworkers [72, 109], initially for applications to on-board GPS systems, are a computational method that allows one to accelerate Dijkstra algorithm for discrete time and state shortest path problems.

The aim of the PhD thesis of Shanqing Liu is to develop new numerical methods to solve Hamilton-Jacobi-Bellman equations that are less sensitive to curse of dimensionality.

In a first work, presented in particular in [20], we have developped a multilevel fast-marching method, extending to the PDE case the idea of "highway hierarchies". Given the problem of finding an optimal trajectory between two given points, the method consists in refining the grid only in a neighborhood of the optimal trajectory, which is itself computed using an approximation of the value function on a coarse grid.

More recently, we have considered general finite horizon deterministic optimal control problems, for which we combine the idea of the multilevel grids around optimal trajectories with the max-plus finite element method solving Hamilton-Jacobi equations introduced in [48].

### 8.1.5 Universal complexity bounds for value iteration applied to zero-sum stochastic games and entropy games

**Participants:** Xavier Allamigeon, Stéphane Gaubert.

In a joint work with Ricardo Katz (CONICET) and Mateusz Skomra (LAAS, CNRS), [23], we have been investigating complexity bounds, based on value iteration, for stochastic zero-sum games problems. This leads for parametrized tractability results for classes of games, including ergodic turn-based stochastic games with a fixed number of random positions, and entropy games. The latter are a class of games in which one player wishes to maximize a topological entropy, whereas the other player wishes to minimize it.

## 8.2 Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

### 8.2.1 Volume in Hilbert and Funk geometries

**Participants:** Constantin Vernicos, Cormac Walsh.

Some of this work is joint work with Dmitry Faifman (Tel Aviv).

In a recent paper [37], we investigated how the volume of a ball in a Hilbert geometry grows as its radius increases. In particular, we studied the volume entropy

$$\lim_{r\to\infty} \frac{\log \mathrm{Vol}\, B(x,r)}{r},$$

where $B(x,r)$ is the metric ball with center $x$ and radius $r$, and Vol denotes the Holmes–Thompson volume. Note that the volume entropy does not depend on the particular choice of $x$. We showed that the volume entropy is exactly twice the flag-approximability of the convex body. This is a new notion of approximability we introduced that measures the complexity of a polytope by counting its number of flags rather than its number of vertices. A corollary is that the Euclidean ball has the maximal volume entropy among Hilbert geometries of a given dimension, a fact that was recently proved by Tholozan by other means. We also showed that the rate of growth of the volume is minimised when the convex body is a simplex.

We are continuing this work by investigating the volume of balls of finite radius, rather than the asymptotics. We have found it convenient to turn our attention to a metric different from the Hilbert metric, but related to it. The Funk metric, as it is called, lacks the symmetry property usually assumed for metric spaces. However, it is somewhat simpler to work with when dealing with volumes, and it exhibits the same interesting behaviour. Given a convex body and a radius $r$, there is a unique point $x$ such that the Funk ball of radius $r$ centered at $x$ minimises the volume over all Funk balls of the same radius. It is natural to conjecture that this minimum volume, which depends on the convex body, it maximised when the body is a Euclidean ball. If this is true, one could recover Blaschke–Santaló inequality by letting the radius tend to zero, and the centro-affine isoperimetric inequality by letting the radius tend to infinity.

Similarly, one could conjecture that, under the assumption that the convex body is centrally symmetric, the minimum volume is minimised when the body is a Hanner polytope. Recall that these are the polytopes that can be constructed from copies of the unit interval by taking products and polar duals. This conjecture interpolates between the Mahler conjecture and Kalai's flag conjecture. We can prove it for unconditional bodies, that is, bodies that are symmetric through reflections in the coordinate hyperplanes.

We have been studying in more detail the volume of balls in the Funk geometry when the convex body is a polytope. Here, as in the case of the Hilbert metric, the volume grows polynomially with order equal to the dimension, and the constant on front of the highest order term depends only on the number of flags of the polytope. Thus, this term does not change when the polytope is perturbed in way that doesn't change the combinatorics. This motivates us to look at the second highest order term. We have developed a formula for this in terms of the position of the vertices of the polytope and the vertices of its dual. Thus we get a new centro-affine invariant for polytopes. We are in the process of studying this invariant, to see where it is maximised and minimised, etc.

We are also studying the following refinement of the volume entropy. We can show that for any Hilbert geometry $K$,

$$\lim_{r\to\infty} \frac{\mathrm{Vol}\, B(x,r)}{e^{(n-1)r}} = \frac{1}{n-1} A_p(K).$$

The limit on the left-hand-side is called the *entropy coefficient.* It can be thought of as the constant on front of the highest order term $e^{(n-1)r}$ in the growth of the volume. The quantity $A_p(K)$ is the *centro-projective area* of the convex body $K$. This quantity can be expressed as an integral over the boundary, and is invariant under projective transformations that leave the origin unchanged. The equation above had previously been established by Berck–Bernig–Vernicos in the case where the boundary of $K$ is $C^{1,1}$, that is, where it is differentiable with Lipschitz continuous derivative. The novelty is that we can now prove it for completely general convex bodies. The new argument builds on work of Tholozan relating the Hilbert metric to the Blaschke metric on the convex body. This allows us to control the contribution to the volume of regions close to the parts of the boundary where the curvature is singular. The argument can also be modified to work for the Funk geometry. Here, the constant that appears on front of the entropy term is the *centro-affine area*, a well-known invariant from convexity theory.

#### 8.2.2 Intrinsic charaterization of Hilbert geometries

**Participants:** Constantin Vernicos.

This is a joint work with Antonin Guilloux (IMJ).

In a Hilbert geometry, if one knows the shape of a metric ball of radius $R$ at some point, then one knows the geometry it lives in. However knowing the tangent unit ball at one point is not enough. In dimension one a simple computation shows that one needs to know the tangent ball at two points. Hence we conjecture that in dimension $n$ one needs to know the shape of tangent ball at $n+1$ points to be able to characterize the convex sets defining the geometry. We were able to prove the conjecture for polytopes in dimension 2, and we are currently looking at smooth convex sets and higher dimension.

### 8.2.3 Reflexion and refraction in Finsler geometry

**Participants:** Constantin Vernicos.

The law of reflection in a smooth strictly convex Finsler geometry was established by Gutkin and Tabachnikov who were studying billards in that setting . Establishing it in the non-smooth not-strictly-convex case forced us to give an affine interpretation which also permits one to establish Snell-Descartes laws of refraction. We also get to give an affine interpretation of refraction with negative indicies. We are investigating existence of periodic orbits in that setting.

### 8.2.4 Barycentres in Funk/Hilbert geometries

**Participants:** Constantin Vernicos.

The existence of certain type of Barycentre in a metric geometry is related to certain types of curvature. For instance in Hyperbolic geometry the fact that the square of the distance is strictly convex gives existence, and is the founding fact in the barycentric methods introduced by Besson-Courtois-Gallot, which allowed them to prove the minimal entropy theorem. We are investigating the existence of such barycentres in the setting of Hilbert geometries, which share some similarity with the hyperbolic geometries, but whose distance is never convex in the sense of Busemman, which is a weaker assumption than what happens in Hyperbolic geometry.

### 8.2.5 Funk and Hilbert geometries of tropical convex sets

**Participants:** Constantin Vernicos, Stéphane Gaubert.

We investigated an analogue of Funk and Hilbert geometries inside tropical convex sets. This is motivated by the construction of barriers adapted to tropical polyhedra, and by nonarchimedean convexity.

### 8.2.6 Firm non-expansive mappings

**Participants:** Armando Gutiérrez, Cormac Walsh.

A fundamental question in the theory of metric spaces is the long term behaviour of iterates of non-expansive mappings. Particularly interesting is the case where the mapping has no fixed point, because here the iterates have the possibility of escaping to infinity.

In [18], we introduce the notion of *firm non-expansive mapping* in an arbitrary (weak) metric space. We show that that, for these mappings, the minimal displacement, the linear rate of escape, and the asymptotic step size are all equal.

This generalises a previous result of Reich and Shafrir in the setting of Banach spaces, and also one by Ariza-Ruiz et al. in the setting of “W -hyperbolic spaces”, which are geodesic metric spaces with a certain “negative curvature”-type condition.

The advantage of our definition is that we do not need to assume the existence of geodesics. This is significant since in modern optimisation applications one often deals with discrete spaces or has access to a collection of points of a space whose geometric structure is unknown. Our class of firm non-expansive mappings includes the mappings considered in the setting of Banach spaces or W-hyperbolic spaces.

Our definition of firm non-expansive was inspired by Ćirić’s work on generalisations of Banach’s Contraction Theorem, for which he introduced the notion of *generalised contraction*. The relation between this concept and our firm non-expansive mappings is similar to the relation between strict contractions and non-expansive mappings.

#### 8.2.7 Metric functionals and invariant spaces

**Participants:** Armando Gutiérrez.

In a joint work with Anders Karlsson (University of Geneva and Uppsala University), we use the explicit formulas for metric functionals on $\ell_1$ and on Hilbert spaces to provide a new result for non-expansive mappings in $\ell_1$ and to study the well-known invariant subspace problem, respectively.

More precisely, the first result is that for every non-expansive mapping $T$ in $\ell_1$, there exists a non-trivial continuous linear functional $f$ such that $f(T^n 0) \geq 0$ for all $n \geq 0$. The second result is that for every affine non-expansive mapping $T = U + T0$ in a Hilbert space, if 0 is not an element of the norm closure of the set Ran(I-T), then there exists a co-dimension one closed invariant subspace for the linear operator $U$.

### 8.3 Tropical algebra and convex geometry

#### 8.3.1 Formalizing convex polyhedra in Coq

**Participants:** Xavier Allamigeon, Quentin Canu.

In a joint work with Ricardo Katz (Conicet, Argentina) and Pierre-Yves Strub (Meta), we present the first formalization of faces of polyhedra in the proof assistant Coq. This builds on the formalization of a library providing the basic constructions and operations over polyhedra, including projections, convex hulls and images under linear maps. Moreover, we design a special mechanism which automatically introduces an appropriate representation of a polyhedron or a face, depending on the context of the proof. We demonstrate the usability of this approach by establishing some of the most important combinatorial properties of faces, namely that they constitute a family of graded atomistic and coatomistic lattices closed under sublattices. We also prove a theorem due to Balinski on the $d$-connectedness of the adjacency graph of polytopes of dimension $d$. This is implemented in the CoqPolyhedra library (we refer to the software session for more details). This work has been published in the journal Logical Methods in Computer Science [14].

In a joint work with Pierre-Yves Strub (Meta), we have achieved the formal verification of a counterexample of Santos et al. to the so-called Hirsch Conjecture on the diameter of polytopes. In contrast with the pen-and-paper proof, our approach is entirely computational: we implement in Coq and prove correct an algorithm that explicitly computes, within the proof assistant, vertex-edge graphs of polytopes as well as their diameter. The originality of this certificate-based algorithm is to achieve a tradeoff between simplicity and efficiency. Simplicity is crucial in obtaining the proof of correctness of the algorithm. This proof splits into the correctness of an abstract algorithm stated over proof-oriented data types and the correspondence with a low-level implementation over computation-oriented data types. A special effort has been made to reduce the algorithm to a small sequence of elementary operations (e.g., matrix multiplications, basic routines on sets and graphs), in order to make the derivation of the correctness of the low-level implementation more transparent. Efficiency allows us to scale up to polytopes with a challenging combinatorics. For instance, we formally check the two counterexamples of Matschke, Santos and Weibel to the Hirsch conjecture, respectively 20- and 23-dimensional polytopes with 36 425 and 73 224 vertices involving rational coefficients with up to 40 digits in their numerator and denominator. We also illustrate the performance of the method by computing the list of vertices or the diameter of well-known classes of polytopes, such as (polars of) cyclic polytopes involved in McMullen's Upper Bound Theorem. This work has been accepted for publication in the proceedings of the conference CPP'23.

#### 8.3.2 Linear algebra over systems

**Participants:** Marianne Akian, Stéphane Gaubert.

In a joint work with Louis Rowen (Univ. Bar Ilan), [27], we study the properties of "systems". The latter provide a general setting encompassing extensions of the tropical semifields and hyperfields. Moreover, they have the advantage to be well adapted to the study of linear or polynomial equations. In particular, in [27], we characterize the semiring systems which arise from hyperrings.

We are now studying linear algebra properties over "systems".

#### 8.3.3 Ambitropical convexity and Shapley retracts

**Participants:** Marianne Akian, Stéphane Gaubert.

Closed tropical convex cones are the most basic examples of modules over the tropical semifield. They coincide with sub-fixed-point sets of Shapley operators – dynamic programming operators of zero-sum games. We study a larger class of cones, which we call "ambitropical" as it includes both tropical cones and their duals. Ambitropical cones can be defined as lattices in the order induced by $\mathbb{R}^n$. Closed ambitropical cones are precisely the fixedpoint sets of Shapley operators. They are characterized by a property of best co-approximation arising from the theory of nonexpansive retracts of normed spaces. Finitely generated ambitropical cones arise when considering Shapley operators of deterministic games with finite action spaces. Moreover, finitely generated ambitropical cones are special polyhedral complexes whose cells are alcoved poyhedra, and locally, they are in bijection with order preserving retracts of the Boolean cube. We also showed that a cone is ambitropical if and only if it is hyperconvex. This is a joint work with Sara Vannucci (Praha). See [39].

#### 8.3.4 Tropical linear regression and applications

**Participants:** Marianne Akian, Stéphane Gaubert, Yang Qi, Omar Saadi.

In [40], we show that the problem consisting in computing a best approximation of a collection of points by a tropical hyperplane is equivalent to solving a mean payoff game, and also, to compute the maximal radius of an inscribed ball in a tropical polytope. We provide an application to a problem of auction theory – measuring the distance to equilibrium. We also study a dual problem — computing the minimal radius of a circumscribed ball to a tropical polytope – and apply it to the rank-one approximation of tropical matrices and tensors.

### 8.3.5 Approximation by small rank tropical tensors

**Participants:** Marianne Akian, Stéphane Gaubert, Jiawei He, Yang Qi.

The max-plus linearity of the dynamic programming operator of a deterministic optimal control problem allows one to develop tropical numerical methods consisting in approximating the value function as the supremum of a finite number of basis functions (see for instance [95, 93, 48]). One can consider in particular the approximation by tropical tensors with small rank, which means that the basis functions are tropical tensors of rank 1, or equivalently sums of functions of one variable only.

During the internship of Jiawei He (of 3rd year of Polytechnique), we have studied and compared experimentally several approximations of functions by tropical tensors of small rank, obtained by sampling or optimization.

### 8.3.6 Roots over the symmetrized tropical semiring and eigenvalues of tropical symmetric matrices

**Participants:** Marianne Akian, Stéphane Gaubert.

The tropical semifield can be thought of as the image of a field with a non-archimedean valuation. It allows in this way to study the asymptotics of Puiseux series with complex coefficients. When dealing with Puiseux series with real coefficients and with its associated order, it is convenient to use the symmetrized tropical semiring introduced in [104] (see also [59]), and the signed valuation which associates to any series its valuation together with its sign.

In a work [28] which started during the postdoc of Hanieh Tavakolipour (Amirkabir University of Technology) in the team, we study the roots's multiplicities and the factorization of polynomials over the symmetrized tropical semiring. We then deduce a Descartes' rule of sign over ordered valued fields. This builds in particular on [60] (for multiplicities) and on [34].

We now study with these tools the asymptotics of eigenvalues and eigenvectors of symmetric positive definite matrices over the field of Puiseux series.

### 8.3.7 Tropical Systems of Polynomial Equations

**Participants:** Marianne Akian, Antoine Bereau, Stéphane Gaubert.

The PhD thesis of Antoine Bereau, started in September 2021, deals with systems of polynomial equations over tropical semifields. We established a nullstellensatz for sparse tropical polynomial systems. We reduce a polynomial system to a linearized system obtained by an appropriate truncation of the Macaulay matrix. Our approach is inspired by a construction of Canny-Emiris (1993), refined by Sturmfels (1994). It leads to an improved estimate of the truncation degree. We also establish a tropical positivstellensatz, allowing one to decide the containment of tropical basic semialgebraic sets. This method leads to the solution of systems of tropical linear equalities and inequalities, which reduces to mean payoff games.

## 8.4 Tropical methods applied to optimization, perturbation theory and matrix analysis

### 8.4.1 Tropicalization of interior point methods and application to complexity

**Participants:** Xavier Allamigeon, Stéphane Gaubert, Nicolas Vandame.

It is an open question to determine if the theory of self-concordant barriers can provide an interior point method with strongly polynomial complexity in linear programming. In the special case of the logarithmic barrier, it was shown in [54],[7] that the answer is negative.

In a subsequent work [41] with Abdellah Aznag (Columbia University) and Yassine Hamdi (Ecole Polytechnique), we have studied the tropicalization of the central path associated with the entropic barrier studied by Bubeck and Eldan (Proc. Mach. Learn. Research, 2015), i.e., the logarithmic limit of this central path for a parametric family of linear programs defined over the field of Puiseux series. Our main result is that the tropicalization of the entropic central path is a piecewise linear curve which coincides with the tropicalization of the logarithmic central path studied by Allamigeon et al. in [54],[7].

In the work [24], we have now shown that *none* of the self-concordant barrier interior point methods is strongly polynomial. This result is obtained by establishing that, on parametric families of convex optimization problems, the log-limit of the central path degenerates to the same piecewise linear curve, independently of the choice of the barrier function. We also provided an improved counter example, with an explicit linear program that falls in the same class as the Klee–Minty counterexample, *i.e.*, a $n$-dimensional combinatorial cube, in which the number of iterations is $2^n$.

A key tool in this work consists of metric inequalities, controlling the convergence of the log-images of semialgebraic sets to a polyhedral complex (their tropicalization), this is the object of a current work, with Mateusz Skomra.

In a joint work [22] with Daniel Dadush, Georg Loho, Bento Natura and László Végh, we establish a natural connection between the complexity of interior point methods and that of the simplex method, and deduce combinatorial bounds on the number of iterations. In more details, we introduce a new polynomial-time path-following interior point method where the number of iterations also admits a combinatorial upper bound $O(2^n n^{1.5} \log n)$ for an $n$-variable linear program in standard form. The number of iterations of our algorithm is at most $O(n^{1.5} \log n)$ times the number of segments of any piecewise linear curve in the wide neighborhood of the central path. In particular, it matches the number of iterations of any path following interior point method up to this polynomial factor. The overall exponential upper bound derives from studying the 'max central path', a piecewise-linear curve with the number of pieces bounded by the total length of $2n$ shadow vertex simplex paths.

### 8.4.2 Tropical Nash equilibria and complementarity problems

**Participants:** Xavier Allamigeon, Stéphane Gaubert.

Linear complementarity programming is a generalization of linear programming which encompasses the computation of Nash equilibria for bimatrix games. While the latter problem is PPAD-complete, we show in [42] that the analogue of this problem in tropical algebra can be solved in polynomial time. Moreover, we prove that the Lemke–Howson algorithm carries over the tropical setting and performs a linear number of pivots in the worst case. A consequence of this result is a new class of (classical) bimatrix games for which Nash equilibria computation can be done in polynomial time. This is joint work with Frédéric Meunier (Cermics, ENPC).

### 8.4.3 Signed Tropicalization of Polars and application to Matrix Cones

**Participants:** Marianne Akian, Xavier Allamigeon, Stéphane Gaubert.

With Sergey Sergeev (U. Birmingham), we study the tropical analogue of the notion of polar of a cone over the symmetrized tropical semiring (see for instance [104, 59]). We characterize in particular the tropical polars of sets of nonnegative tropical vectors, and relate them with images by the nonarchimedean valuation of classical polars over real closed nonarchimedean fields. We study in particular cones of matrices, and optimization problems.

### 8.4.4 Bounds on roots of (multivariate) polynomial systems

**Participants:** Marianne Akian, Stéphane Gaubert.

With Gregorio Malajovich (Univ. Federal Rio de Janeiro (UFRJ), we study the approximation of the solutions of (classical) polynomial systems, using the roots of some associated tropical polynomial systems, or more generally techniques of tropical geometry. In particular, in a work in progress, we generalize the Cauchy and Lagrange bounds to multivariate polynomials.

## 8.5 Algebraic aspects of tensors and neural networks

### 8.5.1 Topology of tropical ranks

**Participants:** Yang Qi.

The primary goal of [35] is to better understand the topological properties of various tensor ranks, which have useful practical implications.

In this ongoing project, we would like to study the corresponding problems for the space of tensors of a fixed Barvinok rank.

### 8.5.2 Approximation theory of neural networks

**Participants:** Yang Qi.

In [36] (joint work with Lek-Heng Lim and Mateusz Michałek) we show that the empirical risk minimization (ERM) problem for neural networks has no solution in general. More precisely, given a training set $s_1, \ldots, s_n \in \mathbb{R}^p$ with corresponding responses $t_1, \ldots, t_n \in \mathbb{R}^q$, fitting a $k$-layer neural network $\nu_\theta : \mathbb{R}^p \to \mathbb{R}^q$ involves estimation of the weights $\theta \in \mathbb{R}^m$ via an ERM:

$$\inf_{\theta \in \mathbb{R}^m} \sum_{i=1}^{n} \| t_i - \nu_\theta(s_i) \|_2^2.$$

We show that even for $k = 2$, this infimum is not attainable in general for common activations like ReLU, hyperbolic tangent, and sigmoid functions. In addition, we show that for smooth activations $\sigma(x) = 1/\big(1 + \exp(-x)\big)$ and $\sigma(x) = \tanh(x)$, such failure to attain an infimum can happen on a positive-measured subset of responses. For the ReLU activation $\sigma(x) = \max(0, x)$, we completely classify cases where the ERM for a best two-layer neural network approximation attains its infimum. In recent applications of neural networks, where overfitting is commonplace, the failure to attain an infimum is avoided by ensuring that the system of equations $t_i = \nu_\theta(s_i)$, $i = 1, \ldots, n$, has a solution. For a two-layer ReLU-activated network, we show when such a system of equations has a solution generically, i.e., when can such a neural network be fitted perfectly with probability one.

## 8.6 Applications

### 8.6.1 Performance evaluation of emergency call centers

**Participants:** Xavier Allamigeon, Marin Boyet, Baptiste Colin, Stéphane Gaubert.

Since 2014, we have been collaborating with Préfecture de Police (Régis Reboul and LcL Stéphane Raclot), more specifically with Brigade de Sapeurs de Pompiers de Paris (BSPP) and Direction de Sécurité de Proximité de l'agglomération parisienne (DSPAP), on the performance evaluation of the new organization (PFAU, "Plate forme d'appels d'urgence") to handle emergency calls to firemen and policemen in the Paris area. We developed analytical models, based on Petri nets with priorities, and fluid limits, see [55], [56], [63]. In 2019, with four students of École polytechnique, Céline Moucer, Julia Escribe, Skandère Sahli and Alban Zammit, we performed case studies, showing the improvement brought by the two level filtering procedure.

Moreover, in 2019, this work has been extended to encompass the handling of health emergency calls, with a new collaboration, involving responsibles from the four services of medical emergency aid of Assistance Publique – Hôpitaux de Paris (APHP), i.e., with SAMU75, 92, 93, 94, in the framework of a project coordinated by Dr. Christophe Leroy from APHP. As part of his PhD work, Marin Boyet have developed Petri net models capturing the characteristic of the centers (CRRA) handling emergency calls the SAMU, in order to make dimensioning recommendations. Following this, we have been strongly solicited by APHP during the pandemic of Covid-19 in order to determine crisis dimensioning of SAMU. Besides, we have initiated a new collaboration, with SAMU69, also on dimensioning.

In parallel, we have further investigated the theoretical properties of timed Petri nets with preselection and priority routing. We represent the behavior of these systems by piecewise affine dynamical systems. We use tools from the theory of nonexpansive mappings to analyze these systems. We establish an equivalence theorem between priority-free fluid timed Petri nets and semi-Markov decision processes, from which we derive the convergence to a periodic regime and the polynomial-time computability of the throughput. More generally, we develop an approach inspired by tropical geometry, characterizing the congestion phases as the cells of a polyhedral complex. These results are illustrated by the application to the performance evaluation of emergency call centers of SAMU in the Paris area. These results have been published in [33].

In [21], we provided explicit formulæ allowing one to compute the time needed by a call center to return to a stationary state after a bulk of calls. This is based on a turnpike-type theorem for Markov decision processes.

These results are also presented in the Phd thesis [65].

### 8.6.2 Optimal control of energy flexibilities in a stochastic environment

**Participants:** Maxime Grangereau, Stéphane Gaubert.

The PhD thesis of Maxime Grangereau [83] has been cosupervised by Emmanuel Gobet (CMAP), Stéphane Gaubert, and Wim van Aackooij (EDF Labs), it dealt with the application of stochastic control methods to the optimization of flexibilities in energy management.

A first series of results concern the application of mean-field control methods to the smart grid [44], including the modelling of storage resources and decentralized aspects [45]. Another contribution concerns a version of the Newton method in stochastic control [46]. A last series of work concern the multistage and stochastic extension of the Optimal Power Flow problem (OPF). We developed semidefinite relaxations, extending the ones which arise in static and deterministic OPF problems. We provided a priori conditions which guarantee the absence of relaxation gap, and also a posteriori methods allowing one to bound this relaxation gap. We applied this approach on examples of grids, with scenario trees representing the random solar power production [17].

### 8.6.3 Optimal pricing of energy contracts

**Participants:** Stéphane Gaubert, Quentin Jacquet.

The PhD thesis of Quentin Jacquet, is cosupervised by Stéphane Gaubert, Clémence Alasseur (EDF Labs), and Wim van Ackooij (EDF Labs). It concerns the application of bilevel programming methods to the pricing of electricity contracts. We investigated in [47] a new model of customer's response, based on a quadratic regularization. We showed that this model has qualitative properties and a realism similar to the classical models based on the logit-response, while being amenable to mathematical programming and polyhedral techniques, and so to exact solutions, via a reduction to quadratic complementary problems. An application to a set of instances representative of French electricity contracts was also developed in [47].

In [25], we developed a model representing the response of a population of customers to dynamic price offers. This lead to a mean-field Markov decision model, with ergodic cost. We showed in particular the optimality of pricing policies characterized by periodic discounts.

The preprint [29] develops a mean field game model to model an incentive pricing scheme, in which agents compete to get the best reward.

# 9 Bilateral contracts and grants with industry

## 9.1 Bilateral contracts with industry

**Participants:** Stéphane Gaubert.

- Optimal pricing of energy and services. Collaboration with Clémence Alasseur and Wim Van Ackooij, from EDF Labs, with the Phd Work of Quentin Jacquet (CIFRE PhD), supervised by Stéphane Gaubert.

# 10 Partnerships and cooperations

## 10.1 International initiatives

### 10.1.1 STIC/MATH/CLIMAT AmSud project

**ARGO**

**Participants:** Marianne Akian, Stéphane Gaubert.

**Title:** Algebraic Real Geometry and Optimization

**Program:** MATH-AmSud

**Duration:** January 1, 2020 – 2022.

**Local supervisor:** Stephane Gaubert

**Partners:**

- Dickenstein (Argentine)
- Grandjean (Brésil)
- Daniilidis (Chili)

**Inria contact:** Stephane Gaubert

**Summary:** Real Algebraic Geometry deals with the study of the sets of real solutions of polynomial equalities (real algebraic varieties) or inequalities (semi algebraic sets). In optimization, one looks for local or global extrema of a real function on a set; problems in which both the function and the set are semi-algebraic play a fundamental role. Hence, Real Algebraic Geometry and Optimization are closely tied, and the advancement in each of the field requires progress in the other. In addition, the use of semi-algebraic techniques in Optimization, and more generally in Applied Mathematics, has been amplified during the last few years, because the most common operations arising in optimization, although destroying smoothness, respect semi-algebraicity. The current project deals with several key problems in Real Algebraic geometry, Polynomial and semi-algebraic Optimization, and with structural results in constrained optimization and nonsmooth dynamics. The project gather teams with complementary specialties, including real algebraic geometry, tropical geometry, symbolic computation, computational optimization, variational analysis, and dynamical systems (gradient flows), allowing to develop new interactions at the interface of pure and applied mathematics.

#### 10.1.2 Participation in other International Programs

**Bilateral project FACCTS with the University of Chicago**

**Participants:** Stéphane Gaubert.

**Title:** Tropical geometry of deep learning

**Partner Institution(s):** University of Chicago (Statistics) and Ecole polytechnique.

**Program:** Bilateral project FACCTS, between the University of Chicago (Statistics) – Lek-Heng Lim– and Ecole polytechnique – Stéphane Gaubert–

**Partners:** Lek-Heng Lim

**Date/Duration:** 2021-2023

### 10.2 International research visitors

#### 10.2.1 Visits of international scientists

**Other international visits to the team**

**Sara Vannucci**

**Status** Postdoctoral Researcher.

**Institution of origin:** Artificial Intelligence Center FEE CTU

**Country:** Czech Republic

**Dates:** February 28, to March 6, 2022.

**Context of the visit:** Collaboration on Ambitropical convexity.

**Mobility program/type of mobility:** research stay.

**Sergey Sergeev**

**Status** Associate Professor.

**Institution of origin:** University of Birmingham

**Country:** United Kingdom

**Dates:** April 4, to April 10, 2022.

**Context of the visit:** Collaboration on tropical optimization.

**Mobility program/type of mobility:** research stay.

**Shmuel Friedland**

**Status** Professor Emeritus

**Institution of origin:** University of Illinois, Chicago

**Country:** US

**Dates:** November 1, to November 6, 2022.

**Context of the visit:** Collaboration on tropical tensors and entropic programming.

**Mobility program/type of mobility:** research stay.

**Gleb Koshevoy**

**Status** Senior Researcher.

**Institution of origin:** Institute of Information Transmission Problems (Kharkevich Institute), Russian Academy of Sciences.

**Country:** Russia.

**Dates:** November 1, 2022 to January 31, 2023.

**Context of the visit:** INRIA Saclay invited professor programme. Gleb Koshevoy is invited to collaborate with the Tropical team, on topics at the interface of algebraic combinatorics, discrete convexity and tropical geometry. The collaboration program covers the study of ambitropical and hyperconvexity, their interactions with $F$-polynomials, and tropical positivstellensatze.

**Mobility program/type of mobility:** research stay.

#### 10.2.2 Visits to international teams

**Research stays abroad**

- M. Akian: visit of Gregorio Malajovich at Univ. Federal Rio de Janeiro (UFRJ) from June 7, to June 14, 2022 (in the framework of ARGO project).
- S. Gaubert: in the framework of the bilateral FACCTS project, visit of Lek-Heng Lim and Shmuel Friedlay, one week, May 2022.
- S. Gaubert: in the framework of ARGO project, visit to Santiago, Chile, Aug 30-Sept-2, 2022: workshop and collaboration with Gregorio Malajovich (UFRJ).
- Y. Qi: obtained a *junior leader position* for the research semester ‘’Algebraic Geometry with Applications to TEnsors and Secants", Institute of Mathematics of the Polish Academy of Sciences, Warsaw, Sep 12 — Dec 16, 2022. In particular, he participated to the following workshops of this semester: Kickoff workshop, Warsaw University, Sep 18 — Sep 23; Tensors from the physics viewpoint, IMPAN, Warsaw, Oct 3 — Oct 7; Geometry of secants, IMPAN, Warsaw, Oct 24 — Oct 28; Algebraic geometry and complexity theory workshop, IMPAN, Warsaw, Nov 14 — Nov 18; Tensors in statistics, optimization and machine learning, IMPAN, Warsaw, Nov 21 — Nov 25.

### 10.3 National initiatives

#### 10.3.1 ANR

**Participants:** Xavier Allamigeon.

- Project ANR JCJC CAPPS ("Combinatorial Analysis of Polytopes and Polyhedral Subdivisions"). Responsable: Arnau Padrol (IMJ-PRG, Sorbonne Université). Partners : IMJ-PRG (Sorbonne Université), INRIA Saclay (Tropical), LIGM (Université Paris-Est Marne-la-Vallée), LIF (Université Aix-Marseille), CERMICS (École Nationale des Ponts et Chaussées), LIX (École Polytechnique).

#### 10.3.2 Programme Gaspard Monge pour l'optimisation, la recherche opérationnelle et leurs interactions avec les sciences des données

**Participants:** Xavier Allamigeon, Stéphane Gaubert.

- Méthodes tropicales pour le dimensionnement de centres d'appels : application à un centre de supervision EDF. Participants : X. Allamigeon S. Gaubert, P. Bendotti (EDF) et T. Triboulet (EDF).

#### 10.3.3 Centre des Hautes Études du Ministère de l'Intérieur

- Project "Optimisation de la performance de centres de traitement d'appels d'urgence en cas d'événements planifiés ou imprévus", coordinated by X. Allamigeon, involving M. Boyet, B. Colin and S. Gaubert.

#### 10.3.4 Joint INRIA & AP-HP Bernoulli lab project: "URGE"

- The project URGE (Analyse des parcours patients aux URgences et optimisation des prises en charGE), started at the fall 2022, in the framework of the joint INRIA & AP-HP Bernoulli lab. The goal of the project is to develop modelling, simulation, performance analysis, and visualization tools, in order to help physicians to optimize the staffing of emergency services. This collaborative project, of four years, involves the Tropical and Aviz teams from INRIA Saclay, the Dyogene team from INRIA Paris, and the Fédération Hospitalo-Universitaire (FHU) / IMPEC Improving Emergency Care, AP-HP / Sorbonne Université / INSERM. The project is led by X. Allamigeon (Tropical) and Y. Yordanov (AP-HP, Saint-Antoine) and involves S. Gaubert, B. Nguyen (Tropical), Ch. Fricker (Dyogene), J.D. Fekete (Aviz).

# 11 Dissemination

## 11.1 Promoting scientific activities

### 11.1.1 Scientific events: organisation

**General chair, scientific chair**

- S. Gaubert is the coordinator of the Gaspard Monge Program for Optimization, Operations Research and their interactions with data sciences (PGMO), a corporate sponsorhip program, operated by Fondation Mathématique Jacques Hadamard, supported by EDF, see Pgmo site.

**Member of the organizing committees**

- X. Allamigeon co-organized the conference Polytopes in Paris and more: Geometry, Combinatorics and Optimization in June 2022 at IMJ, Paris.
- S. Gaubert co-organizes the "Séminaire Parisien d'Optimisation" at Institut Henri Poincaré.
- S. Gaubert co-organized the "Séminaire Français d'Optimisation" (online).

- C. Vernicos is in charge (“responsable thématique”) of the “Structures projectives” action within GDR Platon.
- C. Vernicos is coorganizer of the “Platon” monthly sessions of the French virtual seminar “Groupes et géométries”.

#### 11.1.2 Scientific events: selection

**Chair of conference program committees**

- S. Gaubert, co-chair of the PGMO Days 2022, EDF Labs, 29-30 Nov.

**Member of the conference program committees**

- M. Akian, member of the Scientific Commitee of “Journées SMAI MODE 2022”, May 30, 2022 to June 3, 2022.
- M. Akian, member of the Program Commitee of ISSAC’2023.
- M. Akian, member of the Scientific Commitee of 11th French Biennal of SMAI (SMAI 2023), May 30 to June 3, 2023.
- Stéphane Gaubert, chair of the conference PGMO days 2022, EDF Labs, Palaiseau.

#### 11.1.3 Journal

**Member of the editorial boards**

- Stéphane Gaubert: member of the editorial board of Journal of Dynamics and Games, Linear and Multilinear Algebra, RAIRO, Springer-SMAI book series.

#### 11.1.4 Research administration

**Inria research administration**

- M. Akian: Alternate elected member of Inria’s Scientific Board.
- X. Allamigeon: member of the scientific committee of INRIA Saclay.

**Other research administration**

- X. Allamigeon: elected member of the committee of applied mathematics department of Ecole polytechnique.
- S. Gaubert, member of the “Commission d’appellation” of ENSTA.
- S. Gaubert, member of the committee of applied mathematics department of Ecole polytechnique.
- C. Vernicos: Elected member and deputy-head rank B of CNU, section 25.
- C. Vernicos: Elected member of CVFU of University of Montpellier.

## 11.2 Teaching - Supervision - Juries

### 11.2.1 Teaching

- M. Akian
  - Course “Markov decision processes: dynamic programming and applications” joint between (3rd year of) ENSTA and M2 “Mathématiques et Applications”, U. Paris Saclay, “Optimization”, 30 hours.
- X. Allamigeon
  - Petites classes et encadrement d’enseignements d’approfondissement de Recherche Opérationnelle en troisième année à l’École Polytechnique (programme d’approfondissement de Mathématiques Appliquées) (niveau M1).
  - Cours du M2 “Optimisation” de l’Université Paris Saclay, cours partagé avec Céline Gicquel (LRI, Université Paris Sud).
  - Co-responsabilité du programme d’approfondissement en mathématiques appliquées (troisième année) à l’École Polytechnique.
- A. Bereau
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- A. Bigel
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- S. Gaubert
  - Co-head of the Master “Optimization” of University Paris-Saclay and IPP.
  - Course “Systèmes à Événements Discrets”, option MAREVA, ENSMP.
  - Course “Algèbre tropicale pour le contrôle optimal et les jeux” of “Contrôle, Optimisation et Calcul des Variations” (COCV) of M2 “Mathématiques et Applications” of Sorbonne University and École Polytechnique.
  - Lecture of Operations Research, third year of École Polytechnique. The lectures notes were published as a book [64].
- S. Liu
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.
- N. Vandame
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a “Monitorat”.

### 11.2.2 Supervision

- PhD: Marin Boyet, registered at Univ. Paris Saclay since October 2018, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon. The defense took place on May 25, 2022.
- PhD: Maël Forcier, registered at ENPC since September 2019, thesis supervisor: Vincent Leclère, cosupervision Stéphane Gaubert. The defense took place on December 14, 2022.

- PhD: Tom Ferragut, on the horosphérical products of Gromov-hyperbolic and Busemann convex metric spaces, registered at U. Montpellier, the defense took place in 2022. Supervisor: C. Vernicos, co-supervisor Jérémie Brieussel.
- PhD in progress: Quentin Canu, registered at Univ. Paris Saclay since October 2020, thesis supervisor: Georges Gonthier (INRIA), cosupervision: Xavier Allamigeon and Pierre-Yves Strub (LIX)
- PhD in progress: Shanqing Liu, registered at IPP (EDMH) since September 2020, thesis supervisor, M. Akian, co-supervised by S. Gaubert.
- PhD in progress: Quentin Jacquet, registered at IPP (EDMH) since November 2020, thesis supervisor, S. Gaubert, co-supervised by Clémence Alasseur and Wim van Ackooij.
- PhD in progress: Antoine Bereau, registered at IPP (EDMH) since September 2021, thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian.
- PhD in progress: Nicolas Vandame, registered at IPP (EDMH) since September 2021, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.
- PhD in progress: Amanda Bigel, registered at IPP (EDMH) since September 2022, main thesis supervisors: Cormac Walsh et Constantin Vernicos, thesis supervisor: Stéphane Gaubert.

#### 11.2.3 Juries

- M. Akian
  - Jury (and reviewer) of the PhD thesis of Eloise Berthier (ENS PSL and Inria Paris), October 27, 2022.
  - Jury of the PhD thesis of Cyrille Vessaire (CERMICS), December 16, 2022.
- X. Allamigeon
  - Jury of the PhD of Marin Boyet, Ecole polytechnique, May 25, 2022.
- S. Gaubert
  - Jury of the PhD of Marin Boyet, Ecole polytechnique, May 25, 2022.
  - Jury (and reviewer) of the PhD of Manuel Radons, TU-Berlin, October 26, 2022.
  - Jury of the PhD of Eugenie Marescaux, Ecole polytechnique, November 21, 2022.
  - Jury (and reviewer) of the PhD of Guilherme Espindola-Winck, Université d'Angers, December 9, 2022.
  - Jury of the PhD of Maël Forcier, CERMICS, December 14, 2022.
  - Jury of the PhD of Othmane Jerhaoui, ENSTA, December 15, 2022.

#### 11.2.4 Communications at conferences and seminars

- M. Akian
  - 2nde Journée MAS-MODE 2022, 7 mars 2022, INRIA Paris. Talk: "Jeux stochastiques à somme nulle: ergodicité, complexité et théorie de Perron-Frobenius non linéaire".
  - Séminaire commun MATHRISK / LPSM, Paris Diderot, 7 avril 2022. Talk: "Tropical numerical methods for solving stochastic control problems".
  - Applied Mathematics Colloquium at the Mathematics Institute, Universidade Federal do Rio de Janeiro (UFRJ). Talk: "Tropical geometry, zero-sum games and nonlinear Perron-Frobenius theory".
  - 9th French-Chilean Meeting of Optimization, June 29, July 1st, 2022, Perpignan. Talk: "Tropical linear regression and mean payoff games".

  - 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2022), 12-16 September 2022, Bayreuth, Germany. Session "Hamilton-Jacobi Equations and Mean Field Games", Talk: "Tropical Numerical Methods for Solving Stochastic Control Problems" (work with J.P. Chancelier, L. Pascal, and B. Tran).

- X. Allamigeon:
  - MIND-SoDA joint seminar, Inria Saclay, February 2022. Talk: "Recent progress in the complexity analysis of linear programming".
  - DGeCo seminar, IMJ, Paris, February 2022. Talk: "Tropical complementarity problems and Nash equilibria".
  - STOC 2022: 54th Annual ACM Symposium on Theory of Computing, Rome, Italy, June 20-24, 2022. Talk: "No Self-Concordant Barrier Interior Point Method Is Strongly Polynomial".
- A. Bereau:
  - Combinatorial, Computational, and Applied Algebraic Geometry (CCAAGS-22), June 27-July 1, 2022, Seattle. Poster: "The Nullstellensatz for Sparse Tropical Polynomial Systems".
  - ARGO2022: Workshop on Algebraic Real Geometry and Optimization, Aug 30-Sept-2, 2022, Santiago, Chile. Talk: "The Nullstellensatz for Sparse Tropical Polynomial Systems".
  - Workshop on Solving Polynomial Equations and Applications, CWI, Amsterdam, 5-7 october, 2022. Poster: "The Nullstellensatz for Sparse Tropical Polynomial Systems".
- Q. Canu:
  - DGeCo seminar, IMJ, Paris. April 2022. Talk: "Calcul Polyédral efficace et formellement prouvé".
  - Gallinette seminar, Inria Nantes, December 2022. Talk: "A formal disproof of the Hirsch conjecture".
- M. Forcier:
  - ECSO-CMS, June 2022, Venice: Exact quantization methods for Multistage Stochastic Linear Problems, *best student paper prize*.
  - SMAI-MODE, June 2022, Limoges: Secondary simplex method for two-stage stochastic linear programs with general cost distribution.
  - ROADEF , February 2022, Lyon: Generalized Adaptive Partition-based method for two-stage stochastic linear programs. Slides.
- S. Gaubert:
  - "Polytopes in Paris and more: Geometry, Combinatorics and Optimization", 27-29 June, 2022. Talk: "Ambitropical convexity, Hyperconvexity, and zero-sum games".
  - "ICALP 2022 conference", IRIF, Paris, 4-8 July, 2022. "Universal complexity bounds based on value iteration and application to entropy games".
  - 9th French-Chilean Meeting of Optimization, June 29, July 1st, 2022, Perpignan. Talk: "Solving mean-payoff problems by value iteration: universal complexity bounds and application to entropy games" (work with X. Allamigeon, R. Katz and M. Skomra).
  - ARGO2022: Workshop on Algebraic Real Geometry and Optimization, Aug 30-Sept-2, 2022, Santiago, Chile. Talk: "Convexity, Mean Payoff Games and Nonarchimedean Convex Programming".
- Q. Jacquet:
  - CDC 2022, Cancun, (hybrid conference), "Ergodic control of a heterogeneous population and application to electricity pricing".

- PGMO Days 2022, EDF Labs, "A Rank-Based Reward between a Principal and a Field of Agents: Application to Energy Savings".

- S. Liu:
  - "Journées SMAI MODE 2022", May 30, 2022 to June 3, 2022. Poster: "A Multilevel Fast-Marching Method for the Minimum Time Problem" (work with M. Akian and S. Gaubert).
  - 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2022), 12-16 September 2022, Bayreuth, Germany. Talk: "A Multilevel Fast-Marching Method" (work with M. Akian and S. Gaubert).
- N. Vandame:
  - "PGMO days", Nov. 29-30, 2022, EDF Labs Paris Saclay, "No Self-Concordant Barrier Interior Point Method Is Strongly Polynomial" (work with X. Allamigeon and S. Gaubert).
- Y. Qi:
  - "Tensors from the physics viewpoint", Institute of Mathematics of the Polish Academy of Sciences, Warsaw, October 2022, "Tropical linear regression and low-rank approximation — a first step in tropical data analysis".
  - "Workshop on tensor theory and methods", Lagrange Mathematics and Computing Research Center, Paris, November 2022, "Tropical linear regression and low-rank approximation — a first step in tropical data analysis".
- C. Walsh:
  - Conference "Géométrie, topologie et dynamique en basses dimensions", Sète, 23-27 May 2022. Talk: "Volume growth in Funk geometry".
  - "Annual meeting of the German Mathematical Society", Berlin, 12-16 Sept. 2022. Talk: "Order isomorphisms and antimorphisms in partially-ordered vector spaces".

# 12 Scientific production

## 12.1 Major publications


[1] M. Akian, S. Gaubert and A. Guterman. 'Tropical polyhedra are equivalent to mean payoff games'. In: *Internat. J. Algebra Comput.* 22.1 (2012), pp. 1250001, 43. DOI: 10.1142/S0218196711006674. eprint: 0912.2462. URL: http://dx.doi.org/10.1142/S0218196711006674.

[2] M. Akian, S. Gaubert and R. Nussbaum. 'Uniqueness of the fixed point of nonexpansive semidifferentiable maps'. In: *Transactions of the American Mathematical Society* 368.2 (Feb. 2016). Also arXiv:1201.1536. DOI: 10.1090/S0002-9947-2015-06413-7. URL: https://hal.inria.fr/hal-00783682.

[3] M. Akian, S. Gaubert and R. Bapat. 'Non-archimedean valuations of eigenvalues of matrix polynomials'. In: *Linear Algebra and its Applications* 498 (June 2016). Also arXiv:1601.00438, pp. 592–627. DOI: 10.1016/j.laa.2016.02.036. URL: https://hal.inria.fr/hal-01251803.

[4] M. Akian, S. Gaubert and C. Walsh. 'The max-plus Martin boundary'. In: *Doc. Math.* 14 (2009), pp. 195–240.

[5] X. Allamigeon, P. Benchimol, S. Gaubert and M. Joswig. 'Combinatorial simplex algorithms can solve mean payoff games'. In: *SIAM J. Opt.* 24.4 (2015), pp. 2096–2117. eprint: 1309.5925.

[6] X. Allamigeon, S. Gaubert, E. Goubault, S. Putot and N. Stott. 'A scalable algebraic method to infer quadratic invariants of switched systems'. In: *Proceedings of the International Conference on Embedded Software (EMSOFT)*. Best paper award. The extended version of this conference article appeared in *ACM Trans. Embed. Comput. Syst., 15(4):69:1–69:20, September 2016.* 2015.

[7] X. Allamigeon, P. Benchimol, S. Gaubert and M. Joswig. 'What Tropical Geometry Tells Us about the Complexity of Linear Programming'. In: *SIAM Review* 63.1 (4th Feb. 2021), pp. 123–164. DOI: 10.1137/20M1380211. URL: https://hal.inria.fr/hal-03505719.

[8] J. Bolte, S. Gaubert and G. Vigeral. 'Definable zero-sum stochastic games'. In: *Mathematics of Operations Research* 40.1 (2014). Also arxiv1301.1967, pp. 171–191. DOI: 10.1287/moor.2014.0666. URL: http://dx.doi.org/10.1287/moor.2014.0666.

[9] S. Friedland, S. Gaubert and L. Han. 'Perron–Frobenius theorem for nonnegative multilinear forms and extensions'. In: *Linear Algebra and its Applications* 438.2 (2013), pp. 738–749. DOI: 10.1016/j.laa.2011.02.042. URL: https://hal.inria.fr/hal-00782755.

[10] S. Gaubert and T. Lepoutre. 'Discrete limit and monotonicity properties of the Floquet eigenvalue in an age structured cell division cycle model'. In: *J. Math. Biol.* (2015). DOI: 10.1007/s00285-015-0874-3. eprint: 1301.2151. URL: http://dx.doi.org/10.1007/s00285-015-0874-3.

[11] S. Gaubert and G. Vigeral. 'A maximin characterization of the escape rate of nonexpansive mappings in metrically convex spaces'. In: *Math. Proc. of Cambridge Phil. Soc.* 152 (2012). https://arxiv.org/abs/1012.4765, pp. 341–363. URL: http://dx.doi.org/10.1017/S0305004111000673.

[12] C. Walsh. 'The horofunction boundary and isometry group of the Hilbert geometry'. In: *Handbook of Hilbert Geometry*. Vol. 22. IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society, 2014. URL: https://hal.inria.fr/hal-00782827.

## 12.2 Publications of the year

### International journals

[13] M. Akian, S. Gaubert, Z. Qu and O. Saadi. 'Multiply Accelerated Value Iteration for Non-Symmetric Affine Fixed Point Problems and application to Markov Decision Processes'. In: *SIAM Journal on Matrix Analysis and Applications* 43.1 (2022). DOI: 10.1137/20M1367192. URL: https://hal.inria.fr/hal-03059718.

[14] X. Allamigeon, R. Katz and P.-Y. Strub. 'Formalizing the Face Lattice of Polyhedra'. In: *Logical Methods in Computer Science* Volume 18, Issue 2 (18th May 2022). DOI: 10.46298/lmcs-18(2:10)2022. URL: https://hal.inria.fr/hal-03915661.

[15] P.-C. Aubin-Frankowski and S. Gaubert. 'Tropical reproducing kernels and optimization'. In: *Integral Equations and Operator Theory* (2022). URL: https://hal.archives-ouvertes.fr/hal-03588622.

[16] T. Garaix, S. Gaubert, J. Josse, N. Vayatis and A. Véber. 'Decision-making tools for healthcare structures in times of pandemic'. In: *Anaesthesia Critical Care & Pain Medicine* 41.2 (Mar. 2022), p. 101052. DOI: 10.1016/j.accpm.2022.101052. URL: https://hal.archives-ouvertes.fr/hal-03606559.

[17] M. Grangereau, W. van Ackooij and S. Gaubert. 'Multi-stage Stochastic Alternating Current Optimal Power Flow with Storage: Bounding the Relaxation Gap'. In: *Electric Power Systems Research* 206 (11th Jan. 2022), p. 107774. DOI: 10.1016/j.epsr.2022.107774. URL: https://hal.archives-ouvertes.fr/hal-03236238.

[18] A. W. Gutiérrez and C. Walsh. 'Firm non-expansive mappings in weak metric spaces'. In: *Archiv der Mathematik* 119 (2022), pp. 389–400. URL: https://hal.archives-ouvertes.fr/hal-03479771.

### International peer-reviewed conferences

[19] M. Akian, J.-P. Chancelier, L. Pascal and B. Tran. 'Tropical numerical methods for solving stochastic control problems'. In: MTNS 2022 - 25th International Symposium on Mathematical Theory of Networks and Systems. Bayreuth (DE), Germany, 12th Sept. 2022. URL: https://hal.inria.fr/hal-03944216.

[20] M. Akian, S. Gaubert and S. Liu. 'A multilevel fast-marching method'. In: MTNS 2022 - 25th International Symposium on Mathematical Theory of Networks and Systems. Bayreuth (DE), Germany, 12th Sept. 2022. URL: https://hal.inria.fr/hal-03944192.

[21] X. Allamigeon, M. Boyet and S. Gaubert. 'Computing Transience Bounds of Emergency Call Centers: a Hierarchical Timed Petri Net Approach'. In: PETRI NETS 2022: Application and Theory of Petri Nets and Concurrency. Vol. 13288. PETRI NETS 2022: Application and Theory of Petri Nets and Concurrency, Springer Lecture Notes in Computer Sciences. Bergen, Norway: Springer, 2022, pp. 90–112. DOI: 10.1007/978-3-031-06653-5_5. URL: https://hal.inria.fr/hal-03913405.

[22] X. Allamigeon, D. Dadush, G. Loho, B. Natura and L. Vegh. 'Interior point methods are not worse than Simplex'. In: 2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS). Denver, United States: IEEE, 31st Oct. 2022, pp. 267–277. DOI: 10.1109/FOCS54457.2022.00032. URL: https://hal.inria.fr/hal-03915650.

[23] X. Allamigeon, S. Gaubert, R. D. Katz and M. Skomra. 'Universal Complexity Bounds Based on Value Iteration and Application to Entropy Games'. In: 49th International Colloquium on Automata, Languages, and Programming (ICALP 2022). Paris, France, July 2022. URL: https://hal.laas.fr/hal-03698207.

[24] X. Allamigeon, S. Gaubert and N. Vandame. 'No self-concordant barrier interior point method is strongly polynomial'. In: STOC '22: 54th Annual ACM SIGACT Symposium on Theory of Computing. Rome Italy, France: ACM, 20th June 2022, pp. 515–528. DOI: 10.1145/3519935.3519997. URL: https://hal.inria.fr/hal-03915670.

[25] Q. Jacquet, W. van Ackooij, C. Alasseur and S. Gaubert. 'Ergodic control of a heterogeneous population and application to electricity pricing'. In: IEEE CDC 2022. Cancun, Mexico, 6th Dec. 2022. URL: https://hal.science/hal-03629189.

[26] Q. Jacquet, W. van Ackooij, C. Alasseur and S. Gaubert. 'Une régularisation quadratique pour la tarification de contrats d'électricité'. In: 23ème congrès annuel de la Société Française de Recherche Opérationnelle et d'Aide à la Décision. Villeurbanne - Lyon, France, 23rd Feb. 2022. URL: https://hal.science/hal-03595445.


**Reports & preprints**


[27] M. Akian, S. Gaubert and L. Rowen. *Semiring systems arising from hyperrings*. 30th Sept. 2022. URL: https://hal.inria.fr/hal-03792658.

[28] M. Akian, S. Gaubert and H. Tavakolipour. *Factorization of polynomials over the symmetrized tropical semiring and Descartes' rule of sign over ordered valued fields*. 13th Jan. 2023. URL: https://hal.inria.fr/hal-03941832.

[29] C. Alasseur, E. Bayraktar, R. Dumitrescu and Q. Jacquet. *A Rank-Based Reward between a Principal and a Field of Agents: Application to Energy Savings*. 6th Sept. 2022. URL: https://hal.archives-ouvertes.fr/hal-03770115.

[30] M. Forcier and V. Leclère. *Generalized adaptive partition-based method for two-stage stochastic linear programs : convergence and generalization*. 25th Jan. 2022. URL: https://hal-enpc.archives-ouvertes.fr/hal-03542218.

[31] Q. Jacquet, A. Bialecki, L. E. Ghaoui, S. Gaubert and R. Zorgati. *Entropic Lower Bound of Cardinality for Sparse Optimization*. 28th Nov. 2022. URL: https://hal.archives-ouvertes.fr/hal-03874638.

[32] Q. Jacquet and R. Zorgati. *Tight Bound for Sum of Heterogeneous Random Variables: Application to Chance Constrained Programming*. 22nd Nov. 2022. URL: https://hal.archives-ouvertes.fr/hal-03865441.

## 12.3 Cited publications


[33] X. Allamigeon, M. Boyet and S. Gaubert. 'Piecewise Affine Dynamical Models of Timed Petri Nets – Application to Emergency Call Centers'. In: *Fundamenta Informaticae* 183.3-4 (2021), pp. 169–201. URL: https://hal.archives-ouvertes.fr/hal-02550006.

[34] X. Allamigeon, S. Gaubert and M. Skomra. 'Tropical spectrahedra'. In: *Discrete and Computational Geometry* 63 (Feb. 2020), pp. 507–548. DOI: 10.1007/s00454-020-00176-1. URL: https://hal.inria.fr/hal-01422639.

[35] P. Comon, L. Lek-Heng, Y. Qi and K. Ye. 'Topology of tensor ranks'. In: *Advances in Mathematics* 367 (24th June 2020), p. 107128. DOI: 10.1016/j.aim.2020.107128. URL: https://hal.archives-ouvertes.fr/hal-02361504.

[36] L.-H. Lim, M. Michalek and Y. Qi. 'Best $k$-layer neural network approximations'. In: *Constructive Approximation* (7th June 2021). URL: https://hal.inria.fr/hal-03088287.

[37] C. Vernicos and C. Walsh. 'Flag-approximability of convex bodies and volume growth of Hilbert geometries'. In: *Annales Scientifiques de l'École Normale Supérieure* 54 (2021), pp. 1297–1315. URL: https://hal.archives-ouvertes.fr/hal-01423693.

[38] M. Akian, J.-P. Chancelier and B. Tran. *Tropical Dynamic Programming for Lipschitz Multistage Stochastic Programming.* 13th Dec. 2020. URL: https://hal.inria.fr/hal-03059701.

[39] M. Akian, S. Gaubert and S. Vannucci. *Ambitropical convexity: The geometry of fixed point sets of Shapley operators.* 17th Aug. 2021. URL: https://hal.inria.fr/hal-03504873.

[40] M. Akian, S. Gaubert, Y. Qi and O. Saadi. *Tropical linear regression and mean payoff games: or, how to measure the distance to equilibria.* to appear in SIAM Disc. Math. 21st June 2021. URL: https://hal.inria.fr/hal-03504875.

[41] X. Allamigeon, A. Aznag, S. Gaubert and Y. Hamdi. *The tropicalization of the entropic barrier.* 2020. URL: https://hal.archives-ouvertes.fr/hal-03151661.

[42] X. Allamigeon, S. Gaubert and F. Meunier. *Tropical complementarity problems and Nash equilibria.* v3, Nov. 2022, to appear in SIAM Disc. Math. 2020. URL: https://hal.archives-ouvertes.fr/hal-03151662.

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